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RESEARCH

Quasi-triangle inequality for the Lempert function

Nikolai Nikolov[1,2] · **Pascal J. Thomas**[3]

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Abstract

The (unbounded version of the) Lempert function l_D on a domain $D \subset \mathbb{C}^d$ does not usually satisfy the triangle inequality, but on bounded $\mathcal{C}^2$-smooth strictly pseudoconvex domains, it satisfies a quasi-triangle inequality: $l_D(a, c) \le C(l_D(a, b) + l_D(b, c))$. We show that pseudoconvexity is necessary for this property as soon as D has a $\mathcal{C}^1$-smooth boundary. We also give estimates of the Lempert function and of other invariants in some domains which are models for local situations, and derive some general local bounds depending on the regularity of the boundary of a domain.

Keywords Lempert function · Kobayashi distance · Pseudoconvexity · Quasi-triangle inequality

Mathematics Subject Classification 32F45

1 Introduction

1.1 Invariant functions and triangle inequalities

The most commonly used invariant Finsler metric on the tangent space of a complex manifold M is the Kobayashi–Royden metric, defined as follows using "analytic discs", i.e. holomorphic maps from the unit disc $\mathbb{D}$ to M:

$$\kappa_M(z; X) := \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, M), \varphi(0) = z, \alpha\varphi'(0) = X\}.$$

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✉ Pascal J. Thomas
pascal.thomas@math.univ-toulouse.fr

Nikolai Nikolov
nik@math.bas.bg

1 Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev 8, 1113 Sofia, Bulgaria

2 Faculty of Information Sciences, State University of Library Studies and Information Technologies, Shipchenski prohod 69A, 1574 Sofia, Bulgaria

3 Institut de Mathématiques de Toulouse, UMR5219, CNRS, UPS, Université de Toulouse, 31062 Toulouse Cedex 9, France

It can be seen [1] as the infinitesimal version of the Lempert function, $l_M(z, w) := \tanh^{-1} \ell_M(z, w)$, where

$$\ell_M(z, w) := \inf\{|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D}, M) \text{ with } \varphi(0) = z, \varphi(\alpha) = w\}.$$

Lempert [2, 3] proved that l_M (and thus ℓ_M) satisfies the triangle inequality and defines a distance when M is a bounded convex domain in $\mathbb{C}^n$. This extends to bounded $\mathcal{C}^2$-smooth $\mathbb{C}$-convex domains, as a consequence of the theorem by Jacquet [4] that they can be exhausted by strongly linearly convex domains with $\mathcal{C}^\infty$ boundaries, see also [5, Remark 7.1.21 (b)].

In general, as pointed out in [2], l_M does not satisfy the triangle inequality and the largest pseudodistance not exceeding l_M is $k_M := \inf_{m\in\mathbb{N}} l_M^{(m)} = \lim_{m\to\infty} l_M^{(m)}$, where

$$l_M^{(m)}(z, w) := \inf\{\sum_{j=0}^{m-1} l_M(y_j, y_{j+1}), y_j \in M, y_0 = z, y_m = w\}.$$

We sometimes denote $l_M^{(\infty)} := k_M$. We may define $\ell_M^{(m)}$ by substituting ℓ_M for l_M in the above definition. Note that $k_M \ge \inf_{m\in\mathbb{N}} \ell_M^{(m)} \ge \tanh k_M$ and those three functions are pseudodistances.

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The quantity k_M is called the Kobayashi (pseudo)distance. It turns out that k_M is the integrated form of κ_M, i.e.

$$k_M(z, w) = \inf_\gamma \int_0^1 \kappa_M(\gamma(t); \gamma'(t))dt, \tag{1}$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Good general references about those topics are the monographs [5, 6].

From now on, we restrict attention to the case where M is a domain D in $\mathbb{C}^n$. We say that a non-negative symmetric function L on $D \times D$ satisfies a *Quasi-Triangle Inequality* (QTI) if there exists $C \geq 1$ such that for any $x, y, z \in D$,

$$L(x, z) \leq C\,(L(x, y) + L(y, z))\,. \tag{2}$$

Notice that when l_D satisfies a QTI (resp. the triangle inequality), then ℓ_D satisfies the corresponding inequality.

There are infinitesimal analogues for this. We say that a Finsler pseudometric F satisfies the triangle inequality when $F(z; X + Y) \leq F(z; X) + F(z; Y)$ for any $z \in M$ and $X, Y \in T_z M$, and a *quasi-triangle inequality* if there exists $C > 0$ such that

$$F(z; X + Y) \leq C(F(z; X) + F(z; Y))$$

for any $z \in M$ and $X, Y \in T_z M$.

In general, the Kobayashi–Royden infinitesimal pseudometric κ_M does not satisfy the triangle inequality; the Kobayashi–Buseman infinitesimal pseudometric, given by $\hat{\kappa}_M(z; X) := \inf \kappa_M^{(m)}(z; X)$, where

$$\kappa_M^{(m)}(z; X) := \inf\left\{\sum_{j=1}^m \kappa_M(z; X_j) : X = \sum_{j=1}^m X_j\right\}$$

was devised to supplement this. It turns out that k_M is also the integrated form of $\hat{\kappa}_M$.

Unlike the actual triangle inequality, a QTI is satisfied by ℓ_D for large classes of domains.

Proposition 1 *If $D \subset \mathbb{C}^n$ is a bounded $\mathcal{C}^2$-smooth strictly pseudoconvex domain, then l_D, and therefore ℓ_D, satisfy a QTI.*

This is a corollary of the main result in [7]: in a bounded $\mathcal{C}^2$-smooth strictly pseudoconvex domain, there is a constant $A \geq 1$ such that $k_D \leq l_D \leq Ac_D \leq Ak_D$, where c_D stands for the Carathéodory distance. It is, however, possible to prove the conclusion of Proposition 1 for ℓ_D in a more elementary way, using a natural localization lemma for the Lempert function which will be useful in the rest of the paper (Lemma 5), and we do so in Sect. 2.

There are many examples of domains where l_D satisfies a QTI without satisfying the triangle inequality. For instance, let D be a pseudoconvex balanced domain. It is known that the equality $l_D(0, \cdot) = k_D(0, \cdot)$ is equivalent to the convexity of D (see e.g. [5, Proposition 4.3.10 (b)]). Take now D to be non-convex. If l_D verified the triangle inequality, then it would be equal to k_D everywhere. So one may find $\varepsilon > 0$ and $x_1, x_2, x_3 \in D$ such that $l_D(x_1, x_2) + l_D(x_2, x_3) < l_D(x_1, x_3) - \varepsilon$. Since D is pseudoconvex, we may exhaust it by strictly pseudoconvex domains $(D_j)_{j\geq1}$. Then $l_{D_j}(x_k, x_m) \to l_D(x_k, x_m)$, $1 \leq k < m \leq 3$, (see e.g. [5, Proposition 3.3.5 (a)]) and hence $l_{D_j}(x_1, x_2)+l_{D_j}(x_2, x_3) < l_{D_j}(x_1, x_3)$ for any j large enough. On the other hand, as we have already claimed, D_j satisfies a QTI.

1.2 Main result

Theorem 2 *If $D \subset \mathbb{C}^n$ is a bounded $\mathcal{C}^1$-smooth domain, and ℓ_D or κ_D satisfies a QTI, then D is pseudoconvex.*

The scheme of the proof of Theorem 2 is as follows: if the domain is not pseudoconvex, we can find a point $p \in \partial D$ which is, in a certain sense, strictly pseudoconcave. Near such a point we can find pairs of points $z, w \in D$ such that $\ell_D^{(2)}(z, w) \ll \ell_D(z, w)$ as $z, w \to p$. This is done in Sect. 3 by comparing upper estimates for $\ell_D^{(2)}$ near a pseudoconcave point (in Proposition 7, which does not depend on the smoothness of ∂D) and lower estimates for l_D near any p, when ∂D is merely $\mathcal{C}^1$-smooth (Proposition 8).

To obtain the lower estimates, we study some model domains in $\mathbb{C}^2$. We thought it interesting to study more completely the growth of the Lempert function along the normal to the boundary in those domains, and this is done in Sect. 4. In Sect. 5, we study estimates for another invariant, the Sibony metric, and derive some further consequences of the estimates for model domains.

When the Lempert function ℓ_D satisfies the triangle inequality, it is clearly continuous (and thus l_D as well). The following shows that this is still the case when only a QTI is satisfied, and that it extends to some other invariants.

Corollary 3 *If $D \subset \mathbb{C}^n$ is a bounded $\mathcal{C}^1$-smooth domain, and ℓ_D (resp. κ_D) satisfies a QTI, then for all $m \in \mathbb{N}$, $\ell_D^{(m)}$ and $l_D^{(m)}$ (resp. $\kappa_D^{(m)}$ and $\hat{\kappa}_D$) are continuous functions.*

This corollary follows from Theorem 2 because any bounded pseudoconvex domain D with Lipschitz boundary is hyperconvex, thus taut (see e.g. [5, Remark 3.2.3 (b)]; D is called taut if the space $\mathcal{O}(\mathbb{D}, D)$ is normal), and from the following proposition, the proof of which is given in Sect. 6.

Proposition 4 *(a) For any domain $D \subset \mathbb{C}^n$, $l_D^{(m)}$, $\ell_D^{(m)}$, $\kappa_D^{(m)}$, and $\hat{\kappa}_D$ are upper semicontinuous functions uniformly in $m \in \mathbb{N}$ pointwise.*

(b) For any taut domain $D \subset \mathbb{C}^n$ and any $m \in \mathbb{N}$, the infimum in the definitions of $l_D^{(m)}$, $\ell_D^{(m)}$, and $\kappa_D^{(m)}$ is attained (i.e. there exist $m-1$ intermediate points, resp. m vectors, and corresponding analytic discs realizing the minimum), and these functions, as well as $\hat{\kappa}_D$, are continuous.

1.3 Open questions

(1) The above shows that continuity of l_D the Lempert function can be seen as another relaxation of the condition that ℓ_D satisfies the triangle inequality. For a $\mathcal{C}^1$-smooth bounded domain D, does the continuity of l_D imply pseudoconvexity of D? It would mean that among bounded $\mathcal{C}^1$-smooth domains, continuity of the Lempert function is equivalent to pseudoconvexity (or tautness, or hyperconvexity, since they are all equivalent in this case).

(2) Our methods only work for domains with boundary of at least $\mathcal{C}^1$ regularity. Does the implication still hold for domains with Lipschitz boundary, or even Hölder?
If the boundary is very irregular, there are counterexamples. For instance, let $D = \mathbb{B}^2 \setminus \{0\}$, where $\mathbb{B}^2$ is the unit ball in $\mathbb{C}^2$. Then one can easily show that $\ell_D = \ell_{\mathbb{B}^2}|_D$ and thus satisfies the triangle inequality, while D is clearly not pseudoconvex. In Remark 10, Sect. 4, we mention some examples of the form $D \setminus E$, where E is an "exceptional" closed set; but none of those examples has a boundary locally representable by the graph of a continuous function.

(3) Finally, the main open question is whether the converse of our theorem holds for $\mathcal{C}^1$-smooth bounded domains: if D is pseudoconvex, does ℓ_D satisfy a QTI?

2 Sufficient conditions for a QTI

We first give a Lempert function version of a localization Lemma which goes back to Royden [8] for the case of the Kobayashi–Royden metric. For $A, B \subset D$, let $\ell_D(A, B) := \inf\{\ell_D(z, w) : z \in A, w \in B\}$.

Lemma 5 *Let $V \Subset U$ be two open sets such that $D \cap U$ is nonempty and connected, $D \setminus U \neq \emptyset$; let $z, w \in D \cap V$, then*

$$\ell_D(D \cap V, D \setminus U)\ell_{D\cap U}(z, w) \leq \ell_D(z, w).$$

Proof Let $\varepsilon > 0$, and φ be any holomorphic map from $\mathbb{D}$ to D such that $\varphi(0) = z$, and $\varphi(\alpha) = w$ for some $\alpha < \ell_D(z, w) + \varepsilon$. If for any $\varepsilon > 0$ and any $r < 1$, $\varphi(D(0, r)) \subset D \cap U$, then $\ell_D(z, w) = \ell_{D\cap U}(z, w)$ and we are done. If not, let

$$r_0 := \sup\{r : \varphi(D(0, r)) \subset D \cap U\} < 1.$$

Since $\varphi(0) = z \in V$, we have $r_0 \geq \ell_D(D \cap V, D \setminus U)$. Let $\psi(\zeta) := \varphi(r_0\zeta)$ for $\zeta \in \mathbb{D}$; $\psi(\mathbb{D}) \subset D \cap U$, so $\frac{\alpha}{r_0} \geq \ell_{D\cap U}(z, w)$, thus

$$\ell_D(z, w) + \varepsilon > \alpha \geq \ell_{D\cap U}(z, w)\ell_D(D \cap V, D \setminus U).$$

Since this holds for all $\varepsilon > 0$, we are done.

Proof of Proposition 1 *for ℓ_D*. Suppose the result fails; then there exists for any $k \in \mathbb{N}$ points $a_k, b_k, c_k \in D$ such that

$$\begin{aligned} 1 \geq \ell_D(a_k, c_k) &\geq k\,(\ell_D(a_k, b_k) + \ell_D(b_k, c_k)) \\ &\geq k(\operatorname{diam} D)^{-1}\,(\|a_k - b_k\| + \|b_k - c_k\|)\,, \end{aligned} \tag{3}$$

so passing to a subsequence all three points a_k, b_k, c_k tend to the same $p \in \overline{D}$. If $p \in D$, then in a neighbourhood of p, $l_D(z, w) \asymp \|z - w\|$, and (3) cannot hold when k is large, so $p \in \partial D$.

Since D is strictly pseudoconvex, we may take neighbourhoods $V \Subset U$ of p such that $D \cap U$ is biholomorphic to a convex domain. For k large enough, $a_k, b_k, c_k \in D \cap V$. By Lemma 5, $l_D \asymp l_{D\cap U}$ on $D \cap V$, which contradicts (3) once again for possibly larger k.

3 Necessary condition for a QTI

Proof of Theorem 2 We proceed by contradiction: supposing that D is not pseudoconvex, we will show that there are triples of points which violate any QTI.

Recall that if D is not pseudoconvex, then there exists an affine 2-complex-dimensional subspace P such that $D \cap P$ is not pseudoconvex as a subset of $\mathbb{C}^2$ [9], see also [10]. Choose coordinates such that $D \cap P = D_0 \times \{0\}$, with $D_0 \subset \mathbb{C}^2$.

First we reduce part of our problem to the study of a model domain. □

Lemma 6 *If $D \subset \mathbb{C}^2$ is a bounded non pseudoconvex domain, there exists $p \in \partial D$, U a neighbourhood of p, and Φ a biholomorphism on U such that $\Phi(p) = 0$, $\Phi^{-1}(\overline{\mathbb{D}^2}) \subset U$ and $\Phi^{-1}(G_2) \subset D$, where $G_2 := \{z \in \mathbb{D}^2 : \operatorname{Re} z_1 < |z_2|^2\}$.*

Proof By [11, Theorem 4.1.25], since D is not pseudoconvex, there exists $p \in \partial D$, a neighorhood U_1 of p, and an affine coordinate system in which $p = 0$ and $D \cap U_1 \supset \mathbb{D}^2 \cap \{\operatorname{Re} z_1 + q(z_1, z_2) < 0\}$, where q is a real-valued polynomial, homogeneous of degree 2, with $\frac{\partial^2 q}{\partial z_2 \partial \bar{z}_2}(0) = -1$.

We can write $q(z_1, z_2) = q_1(z_1) + q_2(z_1, z_2) + \operatorname{Re}(\alpha_2 z_2^2) - |z_2|^2$, where the q_j are real-valued and homogeneous of degree 2, $\alpha_2 \in \mathbb{C}$, and

$$|q_2(z_1, z_2)| \leq C_2|z_1||z_2| \leq \frac{C_2^2}{2}|z_1|^2 + \frac{1}{2}|z_2|^2,$$

so that, in a small enough neighbourhood U_2 of 0,

$$q(z_1, z_2) \le C_1|z_1|^2 + \frac{C_2^2}{2}|z_1|^2 + \mathrm{Re}(\alpha_2 z_2^2) - \frac{1}{2}|z_2|^2.$$

Now define a local biholomorphism Φ by

$$(z_1, z_2) = \Phi^{-1}(z_1', z_2') := (z_1' + \alpha_1 {z_1'}^2 - \alpha_2 {z_2'}^2, z_2'),$$

where $\alpha_1 > C_1 + \frac{C_2^2}{2}$. In these new coordinates, the condition $\mathrm{Re} z_1 + q(z_1, z_2) < 0$ is equivalent to $\mathrm{Re} z_1' + \tilde{q}(z_1', z_2') < 0$ with

$$\begin{aligned}\tilde{q}(z_1', z_2') &< (C_1 + \frac{C_2^2}{2})|z_1'|^2 + o(|z_1'|^2 + |z_2'|^2) \\ &\quad + \alpha_1 \mathrm{Re}({z_1'}^2) - \frac{1}{2}|z_2'|^2 \\ &\le (C_1 + \frac{C_2^2}{2} + \alpha_1 + 1)(\mathrm{Re} z_1')^2 - \frac{1}{3}|z_2'|^2,\end{aligned}$$

when $(z_1', z_2') \in U_2$, a small enough neighbourhood of 0. Finally, setting $C_3 := C_1 + \frac{C_2^2}{2} + \alpha_1 + 1$, the condition reduces to $\mathrm{Re} z_1 + C_3(\mathrm{Re} z_1')^2 < \frac{1}{3}|z_2'|^2$, i.e. $\mathrm{Re} z_1 < \frac{1}{3}(1 + C_3(\mathrm{Re} z_1'))^{-1}|z_2'|^2$, which is implied by $\mathrm{Re} z_1 < \frac{1}{4}|z_2'|^2$ when $z \in U_3$, a possibly smaller neighbourhood of 0.

A last linear change of coordinates yields the desired form of G_2.

Proposition 7 *Choose coordinates as in Lemma 6 such that $0 \in \partial D$ with $G_2 \subset D$, for $t \in (0, 1)$, let $p_t := (-t, 0)$.*

Then for $0 < \delta < \varepsilon$ small enough,

$$\ell_D^{(2)}(p_\delta, p_\varepsilon) \le \ell_{G_2}^{(2)}(p_\delta, p_\varepsilon) \le 2\frac{\varepsilon - \delta}{\varepsilon^{1/2}},$$

and $\kappa_D^{(2)}(p_\varepsilon; (1, 0)) \le 2\varepsilon^{-1/2}$.

Proof Let $q_\delta := (-\delta, \frac{\varepsilon-\delta}{\varepsilon^{1/2}})$. Considering the disc $\psi(\zeta) = (-\delta, \zeta)$, we clearly have $\ell_{G_2}(p_\delta, q_\delta) \le \frac{\varepsilon-\delta}{\varepsilon^{1/2}}$.

On the other hand, if we set $\zeta = x + iy$, the disc $\varphi(\zeta) := (-\varepsilon + \varepsilon^{1/2}\zeta, \zeta)$ satisfies

$$\begin{aligned}|\varphi_2(\zeta)|^2 - \mathrm{Re}\varphi_1(\zeta) &= x^2 + y^2 + \varepsilon - \varepsilon^{1/2}x \\ &\ge (x - \frac{1}{2}\varepsilon^{1/2})^2 + \frac{3\varepsilon}{4} > 0,\end{aligned}$$

so for ε small enough, $\varphi(\mathbb{D}) \subset G_2$ and since $\varphi(0) = p_\varepsilon$, $\varphi(\frac{\varepsilon-\delta}{\varepsilon^{1/2}}) = q_\delta$, $\ell_{G_2}(p_\varepsilon, p_\delta) \le \frac{\varepsilon-\delta}{\varepsilon^{1/2}}$.

To obtain the estimate on $\kappa_{G_2}^{(2)}(p_\varepsilon; (1, 0))$, notice that $(1, 0) = (1, \varepsilon^{-1/2}) + (0, -\varepsilon^{-1/2})$. Then using $\varphi'(0) = \varepsilon^{1/2}(1, \varepsilon^{-1/2})$, we see that $\kappa_{G_2}(p_\varepsilon; (1, \varepsilon^{-1/2})) \le \varepsilon^{-1/2}$, and likewise using the map $\psi(\zeta) = (-\varepsilon, \zeta)$, we obtain $\kappa_{G_2}(p_\varepsilon; (0, -\varepsilon^{-1/2})) \le \varepsilon^{-1/2}$.

To obtain the estimate in $D \subset \mathbb{C}^n$, we also denote $p_t = (-t, 0, \ldots, 0) \in \mathbb{C}^n$ by a slight abuse, then with D_0 as in the beginning of this section, $\ell_D^{(2)}(p_\delta, p_\varepsilon) \le \ell_{D_0}^{(2)}(p_\delta, p_\varepsilon) \le 2\frac{\varepsilon-\delta}{\varepsilon^{1/2}}$, by Proposition 7. The computation for $\kappa^{(2)}$ is analogous.

The proof of the theorem will conclude with the following lower estimates on $\ell_D(p_\delta, p_\varepsilon)$ and $\kappa_D(p_\varepsilon; (1, 0))$.

Proposition 8 *Let $D \subset \mathbb{C}^n$ be a bounded $\mathcal{C}^1$-smooth domain, and for any $p \in \partial D$, let $p_t := p + tn_p$, where n_p is the inner normal vector at p. Then for $0 < \delta < \varepsilon$ small enough, $\ell_D(p_\delta, p_\varepsilon)) \gg \frac{\varepsilon-\delta}{\varepsilon^{1/2}}$ and $\kappa_D(p_\varepsilon; (1, 0)) \gg \varepsilon^{-1/2}$ as $\varepsilon \to 0$.*

Proof Using Lemma 5, it will be enough to prove the conclusion of the Proposition for $D \cap \mathbb{D}^n$, so by the contracting property of the Lempert function, it is enough to find a domain G such that $D \cap \mathbb{D}^n \subset G \subset \mathbb{D}^n$ so that the conclusion holds for G.

Represent the boundary locally by $\{\mathrm{Re} z_1 < \Phi(\mathrm{Im} z_1, z')\}$, with $\nabla\Phi(0, 0) = 0$. The regularity of the boundary implies that $\Phi(\mathrm{Im} z_1, z') \le \psi(|\mathrm{Im} z_1| + |z'|)$, with $\psi : [0, \infty) \longrightarrow [0, \infty)$, $\psi(x) = x\psi_1(x)$, $\lim_{x\to 0}\psi_1(x) = 0$. We way take ψ_1 increasing. Localizing, we reduce ourselves to studying $l_G(p_\varepsilon, p_\delta)$, where $G := \{z \in \mathbb{D}^n : \mathrm{Re} z_1 < \psi(|\mathrm{Im} z_1| + \|z'\|)\}$.

Let $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{D} \longrightarrow D$ be a holomorphic map such that $\varphi(0) = p_\varepsilon$, and there is $\alpha > 0$ such that $\varphi(\alpha) = p_\delta$. Write $B_\alpha(\zeta) := \zeta\frac{\zeta-\alpha}{1-\alpha\zeta}$. For $j \ge 2$, $\varphi_j(\zeta) = h_j(\zeta)B_\alpha(\zeta)$, where h_j is a holomorphic function with $\sup_{\mathbb{D}}|h_j| \le 1$. Let $\beta > \alpha$, to be chosen

It follows that $\|(\varphi_2(\zeta), \ldots, \varphi_n(\zeta))\| \le C\beta^2$ when $|\zeta| < \beta$, so

$$\varphi_1(D(0, \beta)) \subset \{z \in \mathbb{D} : \mathrm{Re} z < \psi(|\mathrm{Im} z| + C\beta^2)\} =: P_\beta.$$

Choose β_ε to be the unique number such that $\psi(C\beta_\varepsilon^2) = \varepsilon$. Notice that $\beta_\varepsilon = \varepsilon^{1/2}F(\varepsilon)$, with $\lim_{\varepsilon\to 0} F(\varepsilon) = \infty$. If $\alpha \ge \beta_\varepsilon$, we are done.

If not, define the holomorphic map f on $\mathbb{C}\setminus[\varepsilon, \infty) \supset P_{\beta_\varepsilon}$ by $f(\xi) := -(-\xi + \varepsilon)^{1/2}$, where the argument of the square root is taken in $(-\pi, \pi)$. We then have $f(P_{\beta_\varepsilon}) \subset \{\mathrm{Re} z < 0\}$, and $f \circ \varphi_1(0) = f(-\varepsilon) = -(2\varepsilon)^{1/2}$, $f \circ \varphi_1(\alpha) = f(-\delta) = -(\varepsilon + \delta)^{1/2}$, so that

$$\begin{aligned}\frac{\alpha}{\beta_\varepsilon} &\ge \frac{(2\varepsilon)^{1/2} - (\varepsilon+\delta)^{1/2}}{(2\varepsilon)^{1/2} + (\varepsilon+\delta)^{1/2}} = \frac{\varepsilon - \delta}{((2\varepsilon)^{1/2} + (\varepsilon+\delta)^{1/2})^2} \\ &\ge \frac{\varepsilon - \delta}{8\varepsilon},\end{aligned}$$

so $\alpha \ge \beta_\varepsilon\frac{\varepsilon-\delta}{8\varepsilon} = \frac{1}{8}F(\varepsilon)\frac{\varepsilon-\delta}{\varepsilon^{1/2}}$.

For the infinitesimal analogue, suppose now that $\varphi(0) = p_\varepsilon$ and there is $\lambda \in \mathbb{C}$ such that $\varphi'(0) = \lambda(1, 0, \ldots, 0)$. Notice that $B_0(\zeta) = \zeta^2$, so we have the same inequalities on

φ_j, $j \geq 2$. Choose β_ε, P_{β_ε}, and f as above, and applying the Schwarz Lemma to the map

$$\mathbb{D} \longrightarrow \mathbb{D}, \qquad z \mapsto \frac{f \circ \varphi_1(\beta_\varepsilon z) + (2\varepsilon)^{1/2}}{f \circ \varphi_1(\beta_\varepsilon z) - (2\varepsilon)^{1/2}},$$

we find $1 \geq \beta_\varepsilon |\lambda| \frac{1}{(2\varepsilon)^{1/2}} \frac{1}{(2\varepsilon)^{1/2}} = |\lambda| \varepsilon^{-1/2} F(\varepsilon)$, so that $\kappa_D(p_\varepsilon; (1,0)) \geq \varepsilon^{-1/2} F(\varepsilon)$.

4 Model domains

For $0 < \mu \leq 2$, let us define

$$G_\mu := \{z \in \mathbb{D}^n : \mathrm{Re} z_1 < \sum_{j=2}^n |z_j|^\mu\} \text{ and } \tilde{G}_\mu$$
$$:= \{z \in \mathbb{D}^n : \mathrm{Re} z_1 < |\mathrm{Im} z_1|^\mu + \sum_{j=2}^n |z_j|^\mu\}. \tag{4}$$

We have used the case $n = 2$, $\mu = 2$ above.

Note that since $G_\mu \subset \tilde{G}_\mu$, for any invariant metric or function L, $L_{G_\mu} \geq L_{\tilde{G}_\mu}$, so we will prove the upper estimates for G_μ and the lower estimates for $\tilde{G}_\mu$. We also have $G_\mu \subset G_{\mu'}$, $\tilde{G}_\mu \subset \tilde{G}_{\mu'}$ when $\mu \geq \mu'$, so $\mu \mapsto \ell_{\tilde{G}_\mu}(p_\delta, p_\varepsilon)$, $\mu \mapsto \ell_{G_\mu}(p_\delta, p_\varepsilon)$ are increasing functions of μ. Recall also that $l/\ell \to 1$ as $l \to 0$, so for estimates near 0 it does not matter which version of the Lempert function we are using.

Infinitesimal metrics for points and vectors along the inner normal to the origin for those models (and for more general pseudoconcave domains) have already been studied in [12–15] among others.

In what follows, for $\zeta, \eta \in \mathbb{D}$, we denote by $m_{\mathbb{D}}(\zeta, \eta) := \left|\frac{\zeta - \eta}{1 - \zeta\bar{\eta}}\right|$ the pseudohyperbolic distance (or Möbius function) in the unit disc. Also we use the notation $X_+ := \max(X, 0)$, for $X \in \mathbb{R}$.

Proposition 9 *For $t \in (0,1)$, let $p_t := (-t, 0)$. Let $0 < \delta < \varepsilon \leq \frac{1}{2}$, $0 < \mu \leq 2$, and $2 \leq m \leq \infty$.*

$$\ell_{G_\mu}(p_\delta, p_\varepsilon) \asymp \ell_{\tilde{G}_\mu}(p_\delta, p_\varepsilon) \asymp \frac{(\varepsilon - \delta)}{\varepsilon^{(1 - \frac{1}{2\mu})_+}} \geq \frac{(\varepsilon - \delta)}{\varepsilon^{(1 - \frac{1}{\mu})_+}}$$
$$\asymp \ell^{(m)}_{G_\mu}(p_\delta, p_\varepsilon) \asymp \ell^{(m)}_{\tilde{G}_\mu}(p_\delta, p_\varepsilon).$$

Note that the points $(z_1, z_2) \in \partial G_\mu$ or $\partial \tilde{G}_\mu$ with $0 < |z_2| < 1$, $\mathrm{Im} z_1 \neq 0$, are all $\mathcal{C}^\infty$-smooth strictly pseudoconcave points, so that even when the (quasi) triangle inequality is satisfied for points of the form p_t, which tend to $(0,0)$, we can find counterexamples to the inequality near any of those other boundary points, by Theorem 2.

Proof In all cases, the projection to the first coordinate maps $\tilde{G}_\mu$ to $\mathbb{D}$, so for all $m \in \mathbb{N}^*$, $\ell^{(m)}_{\tilde{G}_\mu}(p_\delta, p_\varepsilon) \geq k_{\mathbb{D}}(-\delta, -\varepsilon) = m_{\mathbb{D}}(-\delta, -\varepsilon) \asymp \varepsilon - \delta$.

The case when $0 < \mu \leq \frac{1}{2}$.

Note that in this case all the estimates reduce to $\varepsilon - \delta$. It will be enough to exhibit a map $\varphi : \mathbb{D} \longrightarrow G_{\frac{1}{2}}$ such that $\varphi(-\varepsilon) = p_\varepsilon$, $\varphi(-\delta) = p_\delta$. Let $\varphi(\zeta) := \left(\zeta, \frac{\zeta + \varepsilon}{1 + \zeta\varepsilon} \frac{\zeta + \delta}{1 + \zeta\delta}, 0, \ldots, 0\right)$. Let $\zeta = x + iy$, $x, y \in \mathbb{R}$.

It is elementary to show that $\left|\frac{\zeta + \varepsilon}{1 + \zeta\varepsilon}\right| \geq \left|\frac{x + \varepsilon}{1 + x\varepsilon}\right|$. So to prove that φ maps the unit disc to $G_{\frac{1}{2}}$, it is enough to show that for $0 \leq x < 1$, we have

$$x^2 < \frac{x + \varepsilon}{1 + x\varepsilon} \frac{x + \delta}{1 + x\delta},$$

which reduces to $(1 - x^2)(\delta\varepsilon(x^2 + 1) + (\delta + \varepsilon)x) > 0$, which completes the proof.

Upper bound for ℓ_{G_μ}.

Consider the map $\varphi : \mathbb{D} \longrightarrow \mathbb{D}^n$ given by

$$\varphi(\zeta) := \left(-\varepsilon + \varepsilon^{1 - \frac{1}{2\mu}} \zeta, B_\alpha(\zeta), 0, \ldots, 0\right), \tag{5}$$

with $\alpha := \frac{(\varepsilon - \delta)}{\varepsilon^{1 - \frac{1}{2\mu}}}$, and $B_\alpha(\zeta) := \zeta \frac{\zeta - \alpha}{1 - \alpha\zeta}$, a finite Blaschke product with zeros at 0 and α. Thus $\varphi(0) = p_\varepsilon$, $\varphi(\alpha) = p_\delta$, so if we can prove that $\varphi(\mathbb{D}) \subset G_\mu$, then $\ell_{G_\mu}(p_\delta, p_\varepsilon) \leq \alpha$, and this finishes the proof.

Let $\zeta = x + iy$, $x, y \in \mathbb{R}$. It is elementary to check that $|\varphi_2(x + iy)| \geq |\varphi_2(x)|$, so to check that $\varphi(\mathbb{D}) \subset G_\mu$, it is enough to prove that for $x \in (-1, +1)$,

$$-\varepsilon + \varepsilon^{1 - \frac{1}{2\mu}} x < \left(x \frac{x - \alpha}{1 - \alpha x}\right)^\mu \text{ i.e. } -1 + \frac{x}{\varepsilon^{\frac{1}{2\mu}}}$$
$$< \left(\frac{x}{\varepsilon^{\frac{1}{2\mu}}} \left(\frac{x}{\varepsilon^{\frac{1}{2\mu}}} - \frac{\alpha}{\varepsilon^{\frac{1}{2\mu}}}\right)\right)^\mu \frac{1}{(1 - \alpha x)^\mu}.$$

The inequality is obvious for $x \leq \varepsilon^{\frac{1}{2\mu}}$, so we only consider $x > \varepsilon^{\frac{1}{2\mu}}$. Then the last factor is larger than 1, so setting $t := \frac{x}{\varepsilon^{\frac{1}{2\mu}}} \in (1, \infty)$, and noticing that $\frac{\alpha}{\varepsilon^{\frac{1}{2\mu}}} = \frac{\varepsilon - \delta}{\varepsilon}$, it is enough to prove that $t - 1 < t^\mu (t - \frac{\varepsilon - \delta}{\varepsilon})^\mu$ for $t > 1$.

Case 1: $\frac{1}{2} < \mu \leq 1$.

Set as new auxiliary variable $\tau := (t - 1)^{\frac{1}{\mu}}$. We need to see that

$$\tau \leq (1 + \tau^\mu)\left(\frac{\delta}{\varepsilon} + \tau^\mu\right) = \frac{\delta}{\varepsilon} + \left(\frac{\delta}{\varepsilon} + 1\right)\tau^\mu + \tau^{2\mu}. \tag{6}$$

For $\tau \leq 1$, $\left(\frac{\delta}{\varepsilon} + 1\right)\tau^\mu \geq \tau^\mu \geq \tau$, and for $\tau \geq 1$, $\tau^{2\mu} \geq \tau$.

Case 2: $1 < \mu$.

Case 2.1: $(1-\frac{1}{2\mu})\varepsilon < \delta < \varepsilon$.

To obtain (6), it is now enough to prove

$$\tau \le (1+\tau^\mu)\left(1-\frac{1}{2\mu}+\tau^\mu\right), \text{ for } \tau > 0. \tag{7}$$

The inequality is trivially verified for $\tau \le 1-\frac{1}{2\mu}$ (and for $\tau \ge 1$). The derivative of the right-hand side is equal to

$$2\mu\tau^{\mu-1}\left(1-\frac{1}{4\mu}+\tau^\mu\right) \ge 2\mu\left(1-\frac{1}{2\mu}\right)^{\mu-1}\left(1-\frac{1}{4\mu}+\left(1-\frac{1}{2\mu}\right)^\mu\right),$$

for $\tau \ge 1-\frac{1}{2\mu}$. Using $\log(1-t) \ge -(2\ln 2)t$ for $0 \le t \le \frac{1}{2}$, we find $\left(1-\frac{1}{2\mu}\right)^\mu \ge \frac{1}{2}$, and finally the right-hand side of the above equation is $\ge \frac{3}{2}\mu - \frac{1}{4} > 1$, so that (7) holds on $(0,\infty)$.

Case 2.2: $0 < \delta \le (1-\frac{1}{2\mu})\varepsilon$.

Here we cannot use the map defined in (5) any longer. The map will depend $\nu := \lceil \mu \rceil := \min\{m \in \mathbb{N} : m \ge \mu\}$.

Define $\varphi : \mathbb{D} \longrightarrow \mathbb{D}^2$ by

$$\varphi(\zeta) := (-\delta - a\zeta^\nu, B_\alpha(\zeta)), \text{ where } a = a_0\varepsilon^{1-\frac{\nu}{2\mu}}, \tag{8}$$

for a_0 small enough, to be chosen depending on μ, $\alpha := \left(\frac{\varepsilon-\delta}{a}\right)^{1/\nu}$. Then for ε small enough we do have $\varphi(\mathbb{D}) \subset \mathbb{D}^2$, and $\varphi(0) = p_\delta$, $\varphi(\alpha) = p_\varepsilon$, and

$$\left(\frac{\varepsilon}{2\mu a}\right)^{1/\nu} = \frac{1}{(2\mu)^{1/\nu}} a_0^{-1/\nu}\varepsilon^{1/2\mu} \le \alpha \le a_0^{-1/\nu}\varepsilon^{1/2\mu}.$$

If we can prove that $\varphi(\mathbb{D}) \subset G_\mu$, then $\ell_{G_\mu}(p_\delta, p_\varepsilon) \le \alpha$ and we are done.

To see that $\mathrm{Re}\varphi_1(\zeta) < |\varphi_2(\zeta)|^\mu$ for $\zeta \in \mathbb{D}$, it is enough to do it when $\mathrm{Re}\zeta^\nu \ge 0$, which implies that $|\cos(\arg\zeta)| \le \cos\frac{\pi}{\nu}$, so $|\mathrm{Im}\zeta| \ge |\zeta|\sin\frac{\pi}{\nu}$. In particular, $|\zeta-\alpha| \ge c_1\max(a_0^{-1/\nu}\varepsilon^{1/2\mu}, |\zeta|)$, with c_1 depending only on μ.

Then it is enough to show that, for $|\zeta| < 1$,

$$a_0\varepsilon^{1-\frac{\nu}{2\mu}}|\zeta|^\nu \le \frac{c_1^\mu}{2^\mu}|\zeta|^\mu \max(a_0^{-1/\nu}\varepsilon^{1/2\mu}, |\zeta|)^\mu. \tag{9}$$

When $|\zeta| \ge a_0^{-1/\nu}\varepsilon^{1/2\mu}$, this reduces to

$$a_0\varepsilon^{1-\frac{\nu}{2\mu}} \le \frac{c_1^\mu}{2^\mu}|\zeta|^{2\mu-\nu},$$

which is enough to check for $|\zeta| = a_0^{-1/\nu}\varepsilon^{1/2\mu}$ since $2\mu-\nu > 0$. For this value of $|\zeta|$, it then reduces to $a_0^{2\mu/\nu} \le \frac{c_1^\mu}{2^\mu}$.

When on the other hand $|\zeta| \le a_0^{-1/\nu}\varepsilon^{1/2\mu}$, (9) reduces to

$$a_0\varepsilon^{1-\frac{\nu}{2\mu}}|\zeta|^{\nu-\mu} \le \frac{c_1^\mu}{2^\mu}a_0^{-\mu/\nu}\varepsilon^{1/2},$$

which is enough to check for $|\zeta| = a_0^{-1/\nu}\varepsilon^{1/2\mu}$ since $\nu-\mu \ge 0$. But this leads to the same inequality as above, and we are done.

Lower bound for $\ell_{\tilde{G}_\mu}$ when $\frac{1}{2} < \mu$.

Let $\varphi = (\varphi_1, \dots, \varphi_n) : \mathbb{D} \longrightarrow G_\mu$ be a holomorphic map such that $\varphi(0) = p_\varepsilon$, and there is $\alpha > 0$ such that $\varphi(\alpha) = p_\delta$. Write $B_\alpha(\zeta) := \zeta\frac{\zeta-\alpha}{1-\alpha\zeta}$. For $j \ge 2$, $\varphi_j(\zeta) = h_j(\zeta)B_\alpha(\zeta)$ where h_j is a holomorphic function with $\sup_{\mathbb{D}}|h_j| \le 1$.

Choose β such that $(2\beta^2)^\mu = \varepsilon$. We may assume $\alpha < \beta$, since the conclusion holds otherwise. For $j \ge 2$ and $|\zeta| \le \beta$, $|\varphi_j(\zeta)| \le \beta\frac{\beta+\alpha}{1+\beta\alpha} \le 2\beta^2$. This implies that $\varphi_1(D(0,\beta)) \subset \{z \in \mathbb{C} : \mathrm{Re}z < C_\mu(\varepsilon + |\mathrm{Im}z_1|^\mu)\} =: P$.

As in the proof of Proposition 8, define a map f on $\mathbb{C} \setminus [\varepsilon,\infty) \supset P$ by $f(\xi) := -(-\xi+\varepsilon)^{1/2}$, where the argument of the square root is taken in $(-\pi,\pi)$. We then have $f(P) \subset \{\mathrm{Re}z < 0\}$, $f\circ\varphi_1(0) = f(-\varepsilon) = -(2\varepsilon)^{1/2}$, $f\circ\varphi_1(\alpha) = f(-\delta) = -(\varepsilon+\delta)^{1/2}$, so that

$$\frac{\alpha}{\beta} \ge \frac{(2\varepsilon)^{1/2}-(\varepsilon+\delta)^{1/2}}{(2\varepsilon)^{1/2}+(\varepsilon+\delta)^{1/2}} = \frac{\varepsilon-\delta}{((2\varepsilon)^{1/2}+(\varepsilon+\delta)^{1/2})^2} \ge \frac{\varepsilon-\delta}{8\varepsilon},$$

so that $\alpha \gtrsim \frac{(\varepsilon-\delta)\beta}{\varepsilon} \asymp (\varepsilon-\delta)\varepsilon^{\frac{1}{2\mu}-1}$.

Upper bound for $\ell^{(2)}_{G_\mu}$ when $\frac{1}{2} < \mu \le 1$.

Let $q := (-\delta, \varepsilon-\delta)$. Considering the disc $\zeta \mapsto (\zeta-\varepsilon, \zeta)$, which maps to G_1 since $-\varepsilon + \mathrm{Re}\zeta < |\zeta|$, we see that $\ell_{G_\mu}(p_\varepsilon, q) \le \varepsilon - \delta$; and considering $\zeta \mapsto (-\delta, \zeta)$, that $\ell_{G_\mu}(p_\delta, q) \le \varepsilon - \delta$.

Upper bound for $\ell^{(2)}_{G_\mu}$ when $1 < \mu$.

To bound $\ell^{(2)}_{G_\mu}(p_\delta, p_\varepsilon)$ from above, we need to see that the disc $\zeta \mapsto (-\varepsilon + C\varepsilon^{1-\frac{1}{\mu}}\zeta, \zeta)$ maps to G_μ when $C > 0$ is small enough. It is enough to check that $F_\mu(x) := x^\mu - C\varepsilon^{1-\frac{1}{\mu}}x + \varepsilon > 0$ for $0 < x < 1$. The function F_μ attains a minimum for $x = x_\mu := \left(\frac{C}{\mu}\right)^{\frac{1}{\mu-1}}\varepsilon^{\frac{1}{\mu}}$, and $F_\mu(x_\mu) = \varepsilon(1 + C^{\frac{\mu}{\mu-1}}(\mu^{\frac{\mu}{1-\mu}} - \mu^{\frac{1}{1-\mu}})) > 0$ for C small enough.

Then we set $q := (-\delta, \frac{\varepsilon-\delta}{C\varepsilon^{1-\frac{1}{\mu}}})$, and the disc defined above shows that $\ell_{G_\mu}(p_\varepsilon, q) \le \frac{\varepsilon-\delta}{C\varepsilon^{1-\frac{1}{\mu}}}$, while as before it is plain that $\ell_{G_\mu}(p_\delta, q) \le \frac{\varepsilon-\delta}{C\varepsilon^{1-\frac{1}{\mu}}}$.

Lower bound for $\ell^{(2)}_{\tilde{G}_\mu}$ when $\frac{1}{2} \le \mu \le 1$.

This is covered by the remark at the outset of the proof that proves that $\ell^{(2)}_{\tilde{G}_\mu}(p_\delta, p_\varepsilon) \gtrsim \varepsilon - \delta$ in all cases for this range of μ.

Lower bound for $\ell^{(2)}_{\tilde{G}_\mu}$ when $1 < \mu$.

For the lower bound, it is enough to bound $k_{\tilde{G}_\mu}(p_\delta, p_\varepsilon)$ from below. Since we will obtain a quantity of the same type as the upper bound, all the $\ell^{(m)}_{\tilde{G}_\mu}$ must have the same order of magnitude.

To simplify notation, we write this proof for $n = 2$.

Using (1), we consider a near-extremal curve $\gamma : [0; 1] \longrightarrow \tilde{G}_\mu$ such that $\gamma(0) = p_\varepsilon$ and $\gamma(1) = p_\delta$. Denote by $\delta_{\tilde{G}_\mu}(z) := \inf\{\|z - w\|, w \in \mathbb{C}^2 \setminus \tilde{G}_\mu\}$, and by $\pi(z)$ a point in $\partial\tilde{G}_\mu$ such that $\|z - \pi(z)\| = \delta_{\tilde{G}_\mu}(z)$. This is a slight abuse of notation, but for $\mu \geq 1$, in a neighbourhood of $(0, 0)$, where all our points will lie, the complement of $\tilde{G}_\mu$ is convex and so $\pi(z)$ is uniquely defined. Finally for z close enough to $\partial\tilde{G}_\mu$, let ν_z stand for the inner normal vector to $\partial\tilde{G}_\mu$ at $\pi(z)$.

Since $k_{\tilde{G}_\mu}(z, w) \gtrsim \|z - w\|$, it is observed that the upper bound tells us that we may assume $\|\gamma(t) - p_\varepsilon\| \lesssim \varepsilon^{\frac{1}{\mu}}$ for $t \in [0; 1]$. It follows that $\|\gamma(t)\| \lesssim \varepsilon^{\frac{1}{\mu}}$, thus $\delta_{\tilde{G}_\mu}(\gamma(t)) \lesssim \varepsilon^{\frac{1}{\mu}}$, thus $\|\pi(\gamma(t))\| \lesssim \varepsilon^{\frac{1}{\mu}}$. Finally the equation of $\partial\tilde{G}_\mu$ implies $0 \leq \mathrm{Re}\pi(\gamma(t))_1 \lesssim \varepsilon$.

Computing the derivative of $|z_2|^\mu + |\mathrm{Im} z_1|^\mu$, we also see that

$$|\mathrm{Im}(\nu_{\gamma(t)})_1| + |(\nu_{\gamma(t)})_2| \lesssim \|\pi(\gamma(t))\|^{\mu-1} \lesssim \varepsilon^{1-\frac{1}{\mu}}. \tag{10}$$

In particular, since $\nu_{\gamma(t)}$ is a unit vector, for ε small enough, $|\mathrm{Re}(\nu_{\gamma(t)})_1| \geq \frac{1}{2}$.

Using the Intermediate Value Theorem, we may consider the restriction of γ to some interval $[a; b] \subset [0; 1]$ such that $\mathrm{Re}\gamma(a)_1 = -\varepsilon$, $\mathrm{Re}\gamma(b)_1 = -\delta$, and $-\varepsilon \leq \mathrm{Re}\gamma(t)_1 \leq -\delta$, for $a \leq t \leq b$. Of course

$$k_{\tilde{G}_\mu}(p_\delta, p_\varepsilon) \geq \int_a^b \kappa_{\tilde{G}_\mu}(\gamma(t); \gamma'(t))dt.$$

From now on we reparametrize this restriction to be defined on $[0; 1]$ and denote it again by γ. Since $-\varepsilon \leq \mathrm{Re}\gamma(t)_1$, $\delta_{\tilde{G}_\mu}(\gamma(t)) \lesssim \varepsilon$.

Consider the positive measure θ on $[0; 1]$ defined by $d\theta(t) = |\mathrm{Re}\gamma'(t)_1|dt$, so that $\theta([0; 1]) \geq \varepsilon - \delta$, and the set

$$A := \Big\{t \in [0; 1] : \left|\gamma'(t)_2(\nu_{\gamma(t)})_2\right| + \left|\mathrm{Im}\gamma'(t)_1 \mathrm{Im}(\nu_{\gamma(t)})_1\right| \leq \frac{1}{2}\left|\mathrm{Re}\gamma'(t)_1 \mathrm{Re}(\nu_{\gamma(t)})_1\right|\Big\}.$$

Case 1. Suppose $\theta(A) \geq \frac{1}{2}\theta([0; 1])$.

We can deduce from [15, Proposition 2.3] that

$$\kappa_{\tilde{G}_\mu}(z; X) \gtrsim \frac{|\langle X, \nu_z\rangle|}{\delta_{\tilde{G}_\mu}(z)^{1-\frac{1}{\mu}}} + \|X\|.$$

For $t \in A$,

$$|\langle\gamma'(t), \nu_{\gamma(t)}\rangle| \geq \frac{1}{2}\left|\mathrm{Re}\gamma'(t)_1 \mathrm{Re}(\nu_{\gamma(t)})_1\right| \geq \frac{1}{4}\left|\mathrm{Re}\gamma'(t)_1\right|,$$

so, since $\delta_{\tilde{G}_\mu}(\gamma(t))^{1-\frac{1}{\mu}} \lesssim \varepsilon^{1-\frac{1}{\mu}}$,

$$k_{\tilde{G}_\mu}(p_\delta, p_\varepsilon) \gtrsim \varepsilon^{\frac{1}{\mu}-1} \int_A |\mathrm{Re}\gamma'(t)_1|dt \gtrsim \varepsilon^{\frac{1}{\mu}-1}(\varepsilon - \delta).$$

Case 2. Suppose $\theta(A) < \frac{1}{2}\theta([0; 1])$.

Then (10) implies that for $t \in [0; 1] \setminus A$,

$$\begin{aligned}\|\gamma'(t)\|\varepsilon^{1-\frac{1}{\mu}} &\gtrsim \left|\gamma'(t)_2(\nu_{\gamma(t)})_2\right| + \left|\mathrm{Im}\gamma'(t)_1 \mathrm{Im}(\nu_{\gamma(t)})_1\right| \\ &\geq \frac{1}{2}\left|\mathrm{Re}\gamma'(t)_1 \mathrm{Re}(\nu_{\gamma(t)})_1\right| \geq \frac{1}{4}\left|\mathrm{Re}\gamma'(t)_1\right|,\end{aligned}$$

so

$$\begin{aligned}k_{\tilde{G}_\mu}(p_\delta, p_\varepsilon) &\gtrsim \int_{[0;1]\setminus A} \|\gamma'(t)\|dt \\ &\gtrsim \varepsilon^{\frac{1}{\mu}-1} \int_{[0;1]\setminus A} \left|\mathrm{Re}\gamma'(t)_1\right| dt \gtrsim \varepsilon^{\frac{1}{\mu}-1}(\varepsilon - \delta).\end{aligned}$$

Remark 10 If $D := \mathbb{B}^n \setminus L$, where L is closed, $L \cap \mathbb{B}^n \neq \emptyset$ and $H_{2n-2}(L) = 0$, with H_{2n-2} being the Hausdorff measure in real dimension $2n - 2$, then it is an immediate consequence of [5, Theorem 3.4.2] that for any $z, w \in D$, $\ell_D(z, w) = \ell_{\mathbb{B}^2}(z, w)$, and as a consequence l_D verifies the triangle inequality. However, D is not pseudoconvex.

On the other hand, if $D := \mathbb{B}^2 \setminus (\mathbb{C} \times \{0\})$, or $D := \mathbb{D} \times \mathbb{D}^*$, then D is pseudoconvex (and we do not know whether a QTI holds there).

5 More estimates for the growth of metrics

5.1 Estimates for the Sibony metric

Recall the definition of the Sibony metric of a domain D: for $p \in D$, $X \in \mathbb{C}^n$,

$$\begin{aligned}S_D(p, X) &:= \sup\Big\{\partial\bar{\partial}u(p)(X, \bar{X})^{1/2} \\ &:= \left(\sum_{i,j=1}^n \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(p) X_i \bar{X}_j\right)^{1/2} : u \in A(p, D)\Big\},\end{aligned}$$

where $\mathbb{D}$ denotes a unit disc in $\mathbb{C}$, and $A(p, D)$ is the set of (plurisubharmonic) functions on D such that $u(p) = 0$, u is $\mathcal{C}^2$ near p, $\log u$ is plurisubharmonic on D, and $0 \leq u \leq 1$ on D.

The Finsler metric S_D satisfies the triangle inequality (see e.g. [13, Lemma 2]).

Proposition 11 *Let $\mu > 1$ and G_μ be as in (4). In general, for $z \in \mathbb{C}^n$, we write $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$. Then $S_{G_\mu}(p_\varepsilon; X) \gtrsim \varepsilon^{\frac{1}{\mu}-1}|X_1| + \|X\|$.*

Proof We follow the method of proof of [13, Proposition 3]. Using the fact that $\{-\varepsilon\} \times \mathbb{D}^{n-1} \subset G_\mu \subset \mathbb{D}^n$, it is easy to see that $S_D(p_\varepsilon; (0, X')) \asymp \|X'\|$. So, using the triangle inequality, it will be enough to prove $S_{G_\mu}(p_\varepsilon; (1, 0, \ldots, 0)) \gtrsim \varepsilon^{\frac{1}{\mu}-1}$ to imply the result.

For $\zeta \in \mathbb{C}$, let $f(\zeta) = \varepsilon^{2/\mu} \left|\frac{\zeta+\varepsilon}{\zeta-\varepsilon}\right|^2$. An elementary computation shows that $\frac{\partial^2 f}{\partial\zeta\partial\bar\zeta}(-\varepsilon) = \frac{1}{4}(\varepsilon^{\frac{1}{\mu}-1})^2$.

Choose c_1 a small enough constant so that $z \in G_\mu$ and $\|z'\| \le c_1\varepsilon^{1/\mu}$ implies $\operatorname{Re} z_1 \le \varepsilon/2$. Let $\alpha > 0$. We will choose $L, L' > 0$ so that e^u is a candidate for the computation of $S_{G_\mu}(p_\varepsilon; (1, 0, \ldots, 0))$, where

$$u(z, w) := \begin{cases} \max\left\{\log\left(f(z_1) + \|z'\|^2\right), \log\left(L\|z'\|^{2+\alpha}\right)\right\} \\ -L', \qquad \|z'\| \le c_1\varepsilon^{1/\mu} \\ \log\left(L\|z'\|^{2+\alpha}\right) - L', \qquad \|z'\| \ge \frac{c_1}{2}\varepsilon^{1/\mu}. \end{cases}$$

Clearly for z close enough to p_ε this coincides with $\log\left(f(z_1) + \|z'\|^2\right)$, so $\left(\frac{\partial^2(e^u)}{\partial z_1\partial\bar z_1}(p_\varepsilon)\right)^{1/2} \gtrsim \varepsilon^{\frac{1}{\mu}-1}$. We need to see that the function is well defined and plurisubharmonic, which will be achieved if $\log\left(L\|z'\|^{2+\alpha}\right) - L' \ge \log\left(f(z_1) + \|z'\|^2\right)$ in the "collar" region $\frac{c_1}{2}\varepsilon^{1/\mu} \le \|z'\| \le c_1\varepsilon^{1/\mu}$.

Notice that in this region $\operatorname{Re} z_1 \le \varepsilon/2$, so $f(z_1) \le c_2\varepsilon^{2/\mu}$, while $\|z'\|^{2+\alpha} \ge c_3\varepsilon^{(2+\alpha)/\mu}$. So it is enough to take $L \ge c_2c_3^{-1}\varepsilon^{-\alpha/\mu}$. Finally, taking L' large enough, we can ensure that $u \le 0$ on G_μ.

5.2 Estimates near a $\mathcal{C}^{1,\gamma}$-smooth point

The following is a consequence of our estimates on model domains.

Proposition 12 *Let $D \subset \mathbb{C}^n$ be a bounded $\mathcal{C}^{1,\gamma}$-smooth domain, for $0 < \gamma \le 1$, and for $p \in \partial D$, let $p_t := p + tn_p$, where n_p is the inner normal vector at p. Then for $0 < \delta < \varepsilon$ small enough, $\ell_D(p_\varepsilon, p_\delta) \gtrsim (\varepsilon - \delta)\varepsilon^{\frac{1}{2(1+\gamma)}-1} \asymp \varepsilon^{\frac{1}{2(1+\gamma)}} - \delta^{\frac{1}{2(1+\gamma)}}$.*

Note that $t = |p_t - p|$ and that, even though p need not be the closest point to p_t, $\lim_{t\to 0, t>0} \frac{|p_t - p|}{\delta_D(p_t)} = 1$.

Proof Let $\mu := 1 + \gamma$. We can follow the beginning of the proof of Proposition 8, this time using $\Phi(\operatorname{Im} z_1, z') \le C(|\operatorname{Im} z_1|^\mu + |z'|^\mu)$, which follows from $\nabla\Phi(0) = 0$ and the regularity hypothesis. So we can reduce ourselves to studying a lower bound for $l_{\tilde G_\mu}$, and Proposition 9 gives the answer.

Conversely, in a pseudoconcave situation with the same smoothness we can give an upper estimates for $\ell_D^{(2)}(p_\varepsilon, p_\delta)$ and $\kappa_D^{(2)}(p_\varepsilon; n_p)$.

Proposition 13 *Let $D = \{\rho < 0\} \subset \mathbb{C}^n$ be a bounded $\mathcal{C}^{m,\gamma}$-smooth domain, for $0 \le \gamma < 1$, $m \in \mathbb{N}^*$, and for $p \in \partial D$, let $p_t := p + tn_p$, where n_p is the inner normal vector at p. Assume that there is a nonzero vector $v \in T_p^{\mathbb{C}}\partial D$ (the complex tangent space to ∂D at p) and $c > 0$ such that $\rho(p + \lambda v) \le -c|\lambda|^{m+\gamma}$.*

Then

(1) for $0 < \delta < \varepsilon$ small enough, $\ell_D(p_\varepsilon, p_\delta) \lesssim (\varepsilon - \delta)\varepsilon^{\frac{1}{(m+\gamma)}-1} \asymp \varepsilon^{\frac{1}{(m+\gamma)}} - \delta^{\frac{1}{(m+\gamma)}}$;

(2) for $0 < \varepsilon$ small enough, $\kappa_D^{(2)}(p_\varepsilon; n_p) \lesssim \varepsilon^{\frac{1}{(m+\gamma)}-1}$.

Proof Let $\mu := m + \gamma$. After an affine change of variables, we may assume that $0 \in \partial D$ and that $G_\mu^- \times \{0\} \subset D$, where $G_\mu^- := \{z \in \mathbb{D}^2 : \operatorname{Re} z_1 < -|\operatorname{Im} z_1| + |z_2|^\mu\}$, with $p_t = (-t, 0)$, $0 < t < 1$. By inclusion, the proof will reduce to the next Lemma.

Lemma 14 *For $\mu \ge 1$ and ε, δ small enough, (1) $\ell_{G_\mu^-}^{(2)}(p_\varepsilon, p_\delta) \lesssim (\varepsilon - \delta)\varepsilon^{\frac{1}{\mu}-1}$, and (2) for $0 < \varepsilon$ small enough, $\kappa_{G_\mu^-}^{(2)}(p_\varepsilon; n_p) \lesssim \varepsilon^{\frac{1}{\mu}-1}$.*

Proof The proof of (1) is analogous to that of Proposition 7.

Let $q_\delta := \left(-\delta, \frac{\varepsilon-\delta}{\varepsilon^{1-1/\mu}\sqrt{2}}\right)$. Clearly $\ell_{G_\mu^-}(q_\delta, p_\delta) \le \frac{\varepsilon-\delta}{\varepsilon^{1-1/\mu}\sqrt{2}}$.

Let $\varphi(\zeta) := (-\varepsilon + \varepsilon^{1-1/\mu}\frac{\zeta}{\sqrt{2}}, \zeta)$. For small ε, $\varphi(\mathbb{D}) \subset \mathbb{D}^2$. On the other hand, $\varphi(0) = p_\varepsilon$, $\varphi\left(\frac{\varepsilon-\delta}{\varepsilon^{1-1/\mu}\sqrt{2}}\right) = q_\delta$. It remains to see that $\varphi(\mathbb{D}) \subset G_\mu^-$ by checking the inequation. Let $\zeta := x + iy$. It is enough to see that for any x, y,

$$\begin{aligned} \varepsilon^{1-1/\mu}(|x| + |y|)2^{-1/2} &\le \varepsilon + (|x| + |y|)^\mu 2^{-\mu/2} \\ &\le \varepsilon + (x^2 + y^2)^{\mu/2}. \end{aligned}$$

The second inequality is elementary. For the first one, let $t = |x| + |y|$ and notice that for $0 \le t \le \varepsilon^{1/\mu}\sqrt{2}$, $\varepsilon^{1-1/\mu}t^\mu 2^{-1/2} \le \varepsilon$; while for $\varepsilon^{1/\mu}\sqrt{2} \le t$, $t^{\mu-1}2^{-\mu/2} \ge \varepsilon^{1-1/\mu}2^{-1/2}$.

To prove (2), write $(1, 0) = (1, \varepsilon^{\frac{1}{\mu}-1}) - \varepsilon^{\frac{1}{\mu}-1}(0, 1)$. Then $\varepsilon^{1-1/\mu}\sqrt{2}\varphi'(0) = (1, \varepsilon^{\frac{1}{\mu}-1})$ so $\kappa_{G_\mu^-}(p_\varepsilon; (1, \varepsilon^{\frac{1}{\mu}-1})) \le \frac{\varepsilon^{\frac{1}{\mu}-1}}{\sqrt{2}}$; and since clearly $\kappa_{G_\mu^-}(p_\varepsilon; (0, 1)) = 1$, the result follows.

6 Appendix

Proof of Proposition 4 We give the proof for l_D, the same arguments work for ℓ_D, *mutatis mutandis.*

(a) Note that l_D is an upper semicontinuous function (see e.g. [5, Proposition 3.1.14]). Since $l_D^{(m)}$ is an infimum of such functions, it is upper semicontinuous too. The uniformity easily follows from the fact that $(l_D^{(m)})_{m\in\mathbb{N}}$ is a pointwise decreasing sequence which is bounded from below.

(b) For $m = 1$ the results follow by tautness of D (see e.g. [5, Propositions 3.2.7 and 3.2.9]). Further, we will consider only the case $m = 2$ to simplify notations. The same proof works for $m \geq 3$ with more intermediate points.

Let $a, b \in D$. We may take a sequence of points $c_k \in D$ such that $l_D(a, c_k) + l_D(c_k, b) \to l_D^{(2)}(a, b)$. Since D is taut, we have the following.

Claim. Given $L > 0$ and $U \Subset D$, there exists $K \Subset D$ such that for any $a' \in U$ and any $c \in D$ for which $l_D(a', c) \leq L$, one has that $c \in K$.

Note that the analogous fact, but for a single point a instead of a relatively compact set U, follows from [5, Proposition 3.2.1]. The same proof with obvious modifications implies the claim.

It follows that, passing to a subsequence if necessary, we may assume that $c_k \to c \in D$. Then $l_D(a, c) + l_D(c, b) = l_D^{(2)}(a, b)$ by continuity of l_D. Since l_D admits extremal discs, so does $l_D^{(2)}$.

Suppose now that $l_D^{(2)}$ is not continuous at $(a, b) \in D \times D$. Since it is upper semicontinuous, we can find $\delta > 0$ and sequences $a_k \to a$ and $b_k \to b$ such that $\lim_{k\to\infty} l_D^{(2)}(a_k, b_k) \leq l_D^{(2)}(a, b) - 3\delta$. Then for each k large enough we may choose a point $c_k \in D$ such that

$$l_D(a_k, c_k) + l_D(c_k, b_k) \leq l_D^{(2)}(a_k, b_k) + \delta \leq l_D^{(2)}(a, b) - \delta.$$

By the claim above, we may assume that $c_k \to c \in D$. Since l_D is continuous, we get the contradiction $l_D^{(2)}(a, b) \leq l_D(a, c) + l_D(c, b) \leq l_D^{(2)}(a, b) - \delta$.

An analogous proof can be written for $\kappa_D^{(m)}$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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