Results Math (2026) 81:122
Online First
© 2026 The Author(s), under exclusive licence to Springer Nature Switzerland AG
https://doi.org/10.1007/s00025-026-02698-1

Results in Mathematics

On a Geometric Extremum Problem for Convex Cones

Oleg Mushkarov and Nikolai Nikolov

Abstract. We study the optimization problem for minimizing the $(n-1)$-volume of the intersection of a convex cone K in $\mathbb{R}^n$ with a hyperplane through a given point, first considered in [5]. We give a geometric characterization of the stationary hyperplanes for this problem when K is a hyperangle which partially answers a question posed in [5]. Moreover, we study the location of the set S of points for which there is a stationary hyperplane as well as the infimum of the $(n-1)$-volumes of cone segments of K cut off by hyperplanes through a given boundary point of K. As a model example we study in detail the non-negative orthant K of $\mathbb{R}^n$. In this case $S = K^o$ and we show that every point of S lies in a unique stationary hyperplane, which we describe in terms of the unique real root of an irrational equation.

Mathematics Subject Classification. 51M16, 52A38, 49K21.

Keywords. Philon line, Convex cone, Simplex.

1. Introduction

Let K be a closed convex cone in $\mathbb{R}^n$ $(n \geq 2)$ with vertex at the origin O and non-empty interior K^o. Assume that K is pointed, that is $K \cap (-K) = \{O\}$. In [5], the following optimization problem is considered:

$(*)$ *Minimize the* $(n-1)$*-volume of the intersection of* K *with a hyperplane* $\mathcal{H} \not\ni O$ *through a given point* A.

Let $(\cdot,\cdot)$ be the standard inner product in $\mathbb{R}^n$. Write $\mathcal{H}$ as $\mathcal{H}(b) = \{x : (b,x) = 1\}$ $(b \neq 0)$ and denote by $V(b)$ the $(n-1)$-volume of $K(b) = K \cap \mathcal{H}(b)$.

The second named author was partially supported by the Bulgarian National Science Fund, Ministry of Education and Science of Bulgaria under contract KP-06-N82/6.

Then $(*)$ means to minimize $V(b)$ under the constraint $(b,a)=1$ $(a=\overrightarrow{OA})$. Note that $V(b)$ is finite exactly when $K(b)$ is compact, i.e. b is an inner point of the dual cone K^* of K. Recall that $K^*=\{y:(y,x)\geq 0,\forall x\in K\}$. Since $K^*\cap(-K)=\{O\}$, we may assume $A\notin(-K)$ and we may consider only vectors $b\in K^+:=(K^*)^o=\{y:(y,x)>0,\forall x\in K\}$ ($K^+\neq\varnothing$ if and only if K is pointed). Call the respective hyperplanes admissible.

The main Result 3.1 in [5] states that b is stationary for V under the constraint $(b,a)=1$, i.e. $D_b(V)=\lambda a$ (λ is the Lagrange multiplier), if and only if

$$\overrightarrow{AH}=n\overrightarrow{AG}, \tag{1}$$

where H is the orthogonal projection of O on $\mathcal{H}(b)$ and G is the centroid of $K(b)$. We call $\mathcal{H}(b)$ a stationary hyperplane for A.

This result extends the classical case $n=2$, where the respective stationary line is called the *Philon line* (also called *Philo's line*; for more details see, e.g., [2,5], and the monograph [4], pp. 39 and 234–238).

Note that (1) implies that any admissible hyperplane $\mathcal{H}$ is stationary exactly for one point from $\mathcal{H}$. On the other hand, denote by S the set of points $A\in\mathbb{R}^n$ for which there is a stationary hyperplane through A. Since $(*)$ has a solution for any $A\in K^o$, we have that $K^o\subset S\subset\mathbb{R}^n\setminus(-K)$. In [5], a general question is posed, namely, to describe the set S and to characterize the respective stationary hyperplanes. In Proposition 4 we recall the answer to this question for $n=2$ and note that the situation changes when $n\geq 3$. We also discuss the infimum problem related to $(*)$ for a boundary point A of K (Proposition 5).

A main purpose of this note is to show that if K is the non-negative orthant $(\mathbb{R}_0^+)^n$, then $S=K^o$ and any point from S lies in a unique stationary hyperplane (which solves $(*)$). Moreover, we describe this hyperplane in terms of the unique real root of an irrational equation (Proposition 6). Applying this result (for $n=2$), we also find the length of the shortest segment cut from this orthant K by a line through a given point $A\in K^o$ (Proposition 8).

2. Geometric Characterization of Stationary Hyperplanes

In this section we give a geometric characterization of the stationary hyperplanes when K is a hyperangle, i.e. a closed convex cone in $\mathbb{R}^n$ spanned by n linearly independent vectors $e_1,e_2,\dots,e_n$. Moreover, we study the location of the set S defined above as well as the infimum of the $(n-1)$-volumes of cone segments of K cut off by admissible hyperplanes through a given boundary point of K.

Let K be a hyperangle in $\mathbb{R}^n$ and $\mathcal{H}=\mathcal{H}(b)$ a hyperplane such that $b\in K^+$. This means that $\mathcal{H}$ intersects any ray $\overrightarrow{Oe_i}$, say at A_i. Let C be the *circumcenter* of the $(n-1)$-simplex $\mathcal{A}=A_1A_2\dots A_n$. For $n\geq 3$, denote by M the *Monge point* of $\mathcal{A}$, i.e. the common point of the hyperplanes through the centroids of the $(n-3)$-dimensional faces of $\mathcal{A}$ which are orthogonal to the opposite edges of $\mathcal{A}$ (see, e.g., [3]). For $A \in \mathcal{H}$ define A' by

$$\overrightarrow{OA'} = \frac{n-1}{2}\overrightarrow{OA}. \tag{2}$$

Proposition 1. *For $n \geq 3$, any two of the following three conditions imply the third one:*

(i) $\mathcal{H}$ is a stationary hyperplane for A;

(ii) $A'A_1 = A'A_2 = \cdots = A'A_n$;

(iii) $H = M$.

Proof. Let A'' be the orthogonal projection of A' on $\mathcal{H}$. Then, by (2),

$$\overrightarrow{A''A} = \frac{n-3}{2}\overrightarrow{AH}. \tag{3}$$

On the other hand, it is a classical result (see, e.g., [3]) that

$$\overrightarrow{CG} = \frac{n-2}{n}\overrightarrow{CM}. \tag{4}$$

The proposition follows by (1), (3), (4), and the fact that (ii) is equivalent to $A'' = C$. $\square$

Remarks. (i) If $n = 2$, (iii) is missing, i.e. $(i) \Leftrightarrow (ii)$; in other words, l is a Philon line for A exactly when G is the midpoint of $[AH]$.

(ii) Recall that a k-simplex Δ ($k \geq 2$) is called *orthocentric* if the altitudes from the vertices to the opposite facets have a common point, called *orthocenter* of the simplex. It turns out that this is equivalent to orthogonality of any edge to the respective $(k-2)$-dimensional face. Note that the orthocenter of Δ coincides with its Monge point (see ,e.g., [3]).

Assume now that $OA_1A_2\ldots A_n$ is an orthocentric n-simplex ($n \geq 3$). Then $A_1A_2\ldots A_n$ is an orthocentric $(n-1)$-simplex and $H = M$. Hence in this case $(i) \Leftrightarrow (ii)$, too.

Proposition 2. *Let K be a hyperangle in $\mathbb{R}^n$.*

(i) If $K^* \subset K$, then $S = K^o$.

(ii) If an extreme ray of K forms acute angles with the others, then there exists a point $A \in \partial K$ with a unique hyperplane solving problem $(*)$; in particular, $K^o \subsetneq S$.

Proof. Let $\mathcal{H}$ be an admissible hyperplane, and H be the orthogonal projection of O on $\mathcal{H}$. For a point $X \in \mathcal{H}$, denote by $(x_1, x_2, \ldots, x_n)$ its *barycentric coordinates* w.r.t. the $(n-1)$-simplex $\mathcal{A}$ defined at the beginning of this section.

(i) The condition $K^* \subset K$ is equivalent to $K^+ \subset K^o$ which means that $H \in \mathcal{A}^o$ for any admissible hyperplane $\mathcal{H}$, i.e. $h_i > 0$ for any $1 \leq i \leq n$.

Let now $\mathcal{H}$ be a stationary hyperplane for A. Then (1) reads as

$$h_i + (n-1)a_i = 1, \quad 1 \leq i \leq n. \tag{5}$$

Hence $a_i < 1/(n-1)$, and then

$$a_k = 1 - \sum_{i \neq k}^{n} a_i > 1 - (n-1).\frac{1}{n-1} = 0.$$

This shows $A \in K^o$, and thus $S \subset K^o$.

(ii) We may assume $(e_1, e_i) > 0$ for any $2 \leq i \leq n$. Denote by $\mathcal{A}_1 = A_1 A_2 \dots A_n$ the respective $(n-1)$-simplex to the hyperplane $\mathcal{H}_1 = \mathcal{H}(e_1)$. Consider the point $A \in \partial K$ with barycentric coordinates $(0, \frac{1}{n-1}, \dots, \frac{1}{n-1})$ w.r.t. $\mathcal{A}_1$.

Let $\mathcal{H}$ be an admissible hyperplane for A and $\mathcal{A}' = A_1' A_2' \dots A_n'$ be the orthogonal projection of the respective $(n-1)$-simplex $\mathcal{A}$ on H_1. Then $A_1' = A_1$ and A_i' lies on the ray $A_1 A_i$, $2 \leq i \leq n$. Denote by K_1 the cone spanned by these $n-1$ rays.

Recall now (see, e.g., [5, Section 4, item (iv)]) that if the $(n-1)$-volume of the cone segment of K_1 cut off by an $(n-2)$-plane through A is minimal, then A is the centroid of the respective $(n-2)$-simplex. Note that $A_2 \dots A_n$ is the unique $(n-2)$-simplex with this property. So

$$V_{n-1}(\mathcal{A}) \geq V_{n-1}(\mathcal{A}') \geq V_{n-1}(\mathcal{A}_1)$$

and (ii) easily follows. $\square$

It is easy to see that $K \subset K^*$ (resp. $K \subset K^+$) if and only if the angles between the extreme rays of K are non-obtuse (resp. acute). Therefore, if $K \subset K^+$, then the condition in (ii) of Proposition 2 is satisfied for any extreme ray of K. On the other hand, since $(K^*)^* = K$, we have that $K^* \subset K$ if and only if the angles between the inward normals to the facets of K are non-obtuse. In particular, the non-negative orthant is self-dual, i.e. $K = K^*$. Note also that in general $K^o \cap K^+ \neq \varnothing$ which follows by the hyperplane separation theorem.

Next, we claim that if K is a closed trihedral angle in $\mathbb{R}^3$ with face angles $\alpha \leq \beta \leq \gamma$, then the law of cosines for K shows that its dihedral angles are non-acute, i.e. $K^* \subset K$, if and only if $\beta \geq \pi/2$ and $\cos\alpha \leq \cos\beta\cos\gamma$. In particular, $\alpha \geq \pi/2$ implies $K^* \subset K$. This observation remains true in any dimension.

Proposition 3. *If the angles between the extreme rays of a hyperangle K in $\mathbb{R}^n$ are non-acute, then $K^* \subset K$.*

Proof. Suppose that $K^* \not\subset K$. Then, since $K \not\subset K^+$ and $K \cap K^+ \neq \varnothing$, there exists a point $A \in K^+ \cap \partial K$. We may assume that $a = \sum_{i=2}^n \alpha_i e_i$, where $\alpha_i \geq 0$. Hence $0 < (a, e_1) = \sum_{i=2}^n \alpha_i (e_i, e_1) \leq 0$, a contradiction. $\square$

Now we will describe the set S if $n = 2$. Set $\alpha = \angle(e_1, e_2)$. Then $K^* \subset K$ if $\alpha \geq 90^\circ$ and $K \subset K^+$ if $\alpha < 90^\circ$. So Proposition 1 implies that $S = K^o$ in the first case and $K^o \subsetneq S$ in the second one. In fact, we know much more (see, e.g., [1] and [2]).

If $\alpha < 90°$, set

$$\theta = \arctan\left(\frac{1+\sin^2\alpha/2}{1+\cos^2\alpha/2}\right)^{3/2}$$

and consider the closed angle T with vertex O, the same bisector as K, and aperture 2θ. Then $K \subsetneq T$.

Proposition 4. *(i) If $\alpha \geq 90°$, then $S = K^0$, and for any point from K^0 there exists one stationary line (global minima point).*

(ii) If $\alpha < 90$, then $S = T \setminus \{O\}$. Moreover,

– for $A \in K$ there exists one stationary line (global minima point);

– for $A \in T^o \setminus K$ there exist two stationary lines (local minima and maxima points);

– for $A \in \partial T \setminus \{O\}$ there exists one stationary line (saddle point).

It is noted in [5, Section 4 (ii)] that the situation changes when $n \geq 3$. For example, if K is the closed trihedral angle spanned by the vectors $e_1 = (1,-1,-0.2), e_2 = (1,1,-0.2)$, and $e_3 = (1,0,0.01)$ $(K \subset K^+)$, then three stationary planes exist for the point $A = (1,0,0) \in K^o$ (two local minima points and one saddle point).

Finally, let us mention again that if K is a closed convex cone in $\mathbb{R}^n$ with vertex at the origin O, then the problem $(*)$ makes sense if K is pointed, $K^o \neq \varnothing$, and $A \notin (-K)$. It has solution if $A \in K^o$ and it has no solution if $A \notin K \cup (-K)$, since in this case the infimum m_A of the $(n-1)$-volumes of cone segments of K cut off by admissible hyperplanes through A is equal to 0. So, it is natural to ask what happens if $A \in \partial K$. The answer is given by the following proposition which is more or less intuitively clear.

Proposition 5. *Let $A \in \partial K \setminus \{O\}$.*

(i) If there is no hyperplane $\mathcal{T} \ni O$ such that A is a point from the relative interior of $K' = \partial K \cap \mathcal{T}$, then $m_A = 0$.

(ii) Otherwise, such a $\mathcal{T}$ is unique and $m_A > 0$. If m_A is not attained, then m_A is equal to the minimal $(n-1)$-volume m of cone segment K'' of K' cut off by an $(n-2)$-plane $\mathcal{T}' \subset \mathcal{T}$ through A. Moreover, the relation (1) remains true in this degenerate case.

Proof. For any $A \in \partial K \setminus \{O\}$ there is a maximal positive integer $k \leq n-1$ and a unique k-plane $\mathcal{T} \ni O$ such that A is a point from the relative interior of the cone $K' = \partial K \cap \mathcal{T}$. Let $\mathcal{T}' \subset \mathcal{T}$ be a $(k-1)$-plane through A that minimizes the $(k-1)$-volume of the cone segment K'' of K' cut off by $\mathcal{T}'$. Take $n-k$ extreme rays $O\vec{e_1}, \dots, O\vec{e_{n-k}}$ of K not lying in $\mathcal{T}'$ and points $X_{1,i} \in O\vec{e_1}, \dots, X_{n-k,i} \in O\vec{e_{n-k}}$ which tend to O as $i \to \infty$ .These points together with $\mathcal{T}'$ determine admissible hyperplanes $\mathcal{H}_i$ for K and it is not difficult to see that $V_{n-1}(K \cap \mathcal{H}_i) \to V_{n-1}(K'')$. Hence $m_A = 0$ for $k < n-1$ and $m \geq m_A$ for $k = n-1$.

Let $k = n - 1$. It turns out that A is the centroid of $K' \cap \mathcal{T}'$ [5, Section (iv)]). Then $\overrightarrow{AO} = n\overrightarrow{AG}$, where G is the centroid of K'', i.e. (1) holds ($H = O$).

It remains to show that if $k = n - 1$ and m_A is not attained, then $m_A \geq m$. For this, consider a sequence of admissible hyperplanes $\mathcal{H}_i \ni A$ such that $V_{n-1}(K \cap \mathcal{T}_i) \to m_A$.

Claim. $\mathcal{T}_i \to \mathcal{T}$.

Proof of the claim. Let $\mathcal{T}_i = \mathcal{T}_i(b_i) = \{x : (b_i, x) = 1\}$. Since m_A is not attained, then $||b_i|| \to \infty$. On the other hand, $A \in (K')^o$ and therefore there are $n - 1$ linearly independent vectors $e_1, e_2, \ldots, e_{n-1} \in K'$ such that $a = \sum_{k=1}^{n-1} \alpha_k e_k$, where $\alpha_k > 0$. Let $c_i = \frac{b_i}{||b_i||}$. Since $(c_i, e_k) > 0$ and $(c_i, a) = \frac{1}{||b_i||} \to 0$, it follows that $(c_i, e_k) \to 0$ for any $1 \leq k \leq n - 1$. This means that c_i tends to a normal vector of $\mathcal{T}$, i.e. $\mathcal{T}_i \to \mathcal{T}$. $\square$

Further on, denote by Π_i the orthogonal projection of $K \cap \mathcal{T}_i$ on $\mathcal{T}$. Consider a point $O_i \in \Pi_i$ at minimal distance r_i from O. Let K_i be the convex hull of O_i and $B_i = K' \cap \mathcal{T}_i$. Then

$$V_{n-1}(K \cap \mathcal{T}_i) \geq V_{n-1}(\Pi_i) \geq V_{n-1}(K_i). \tag{6}$$

Set $h_i = \operatorname{dist}(O_i, B_i)$, $d_i = \operatorname{dist}(O, B_i)$, and note that

$$\frac{V_{n-1}(K_i)}{V_{n-1}(K_i'')} = \frac{h_i}{d_i}, \tag{7}$$

where K_i'' is the cone segment of K' cut off by the $(n-2)$-plane $\mathcal{T}_i \cap \mathcal{T}$. Now a similar argument to that in the proof of the claim provides by contradiction a $d > 0$ such that $d_i \geq d$.[1] On the other hand, $\mathcal{T}_i \to \mathcal{T}$ easily implies that $r_i \to 0$, and hence

$$\left| \frac{h_i}{d_i} - 1 \right| \leq \frac{r_i}{d_i} \to 0. \tag{8}$$

Since $V_{n-1}(K \cap \mathcal{T}_i) \to m_A$, then (6), (7), (8), and $V_{n-1}(K_i'') \geq m$ imply that $m_A \geq m$. $\square$

3. The Non-Negative Orthant

Let K be the non-negative orthant in $\mathbb{R}^n$, i.e. $K = (\mathbb{R}_0^+)^n$. We already know that $S = K^o$. For $A = (a_1, a_2, \ldots, a_n) \in K^o$, set $b_i = \frac{n-1}{2} a_i$, $1 \leq i \leq n$, and

$$f_A(x) = \sum_{i=1}^{n} \frac{b_i}{b_i + \sqrt{b_i^2 + x}} - \frac{n-1}{2}.$$

[1] In fact, such an argument shows that if A is an interior point of a closed convex pointed cone L, then there exists a $d_A > 0$ such that $\operatorname{dist}(O, \mathcal{H}) > 0$ for any admissible hyperplane $\mathcal{H} \ni A$.

This function is continuous and strictly decreasing on the interval $[-\min b_i^2, +\infty)$, $f(0) = \frac{1}{2}$ and $f(+\infty) = \frac{1-n}{2}$. Hence the equation $f_A(x) = 0$ has a unique real root λ_A, and $\lambda_A > 0$. Set $c_i = b_i + \sqrt{b_i^2 + \lambda_A}$, $1 \le i \le n$.

Proposition 6. *For any $A \in K^o$ there exists a unique stationary hyperplane. For the respective $(n-1)$-simplex $\mathcal{A}$ we have that $\overrightarrow{OA_i} = c_i e_i$, $1 \le i \le n$, and*

$$V_{n-1}(\mathcal{A}) = \frac{\prod_{i=1}^n c_i}{(n-1)!\sqrt{\lambda_A}}.$$

Proof. Let $\mathcal{H}$ be an admissible hyperplane. Then, by Remark (ii) after Proposition 1, $\mathcal{H}$ is a stationary hyperplane for A if and only if $A'A_1 = A'A_2 = \cdots = A'A_n$, where A' is given by (2). Let $\overrightarrow{OA_i} = x_i e_i$, $1 \le i \le n$. Then we get that

$$x_i^2 - (n-1)x_i a_i = A'A_i^2 - A'O^2 =: \lambda, \quad 1 \le i \le n, \tag{9}$$

i.e.

$$1 - (n-1)\frac{a_i}{x_i} = \frac{\lambda}{x_i^2}.$$

Summing up these equalities and using that

$$\sum_{i=1}^n \frac{a_i}{x_i} = 1 \tag{10}$$

(since $A, A_1, A_2, \dots, A_n \in \mathcal{H}$), we get that $\lambda \sum_{i=1}^n x_i^{-2} = 1$, i.e. $\lambda = d^2$, where d is the distance from O to $\mathcal{H}$. By (9), $x_i = b_i + \sqrt{b_i^2 + \lambda}$, and then (10) implies that $f_A(\lambda) = 0$. So $\lambda = \lambda_A$, $x_i = c_i$, and the formula for $V_{n-1}(\mathcal{A})$ follows. □

Let us note that the circumcenter C of the simplex $\mathcal{A}$ coincides with A if only if $n = 3$.

Applying Proposition 6 for $n = 2$ gives the following known fact.

Corollary 7. *The length of a shortest segment cut from the first quadrant in $\mathbb{R}^2$ by a line through its interior point $A(a_1, a_2)$ is equal to $(a_1^{2/3} + a_2^{2/3})^{3/2}$, and such a segment is unique.*

Proof. Let $\lambda = (a_1 a_2)^{2/3}(a_1^{2/3} + a_2^{2/3})$. Since

$$b_1^2 + \lambda = a_1^{2/3}(a_1^{2/3}/2 + a_2^{2/3})^2, \quad b_2^2 + \lambda = a_2^{2/3}(a_2^{2/3}/2 + a_1^{2/3})^2,$$

it easily follows that $\lambda = \lambda_A$. Then $c_1 = a_1 + a_1^{1/3}a_2^{2/3}$, $c_2 = a_2 + a_2^{1/3}a_1^{2/3}$, and hence $V_1 = (c_1^2 + c_2^2)^{1/2} = (a_1^{2/3} + a_2^{2/3})^{3/2}$. □

We now generalize the above corollary to higher dimensions.

Proposition 8. *Let $A = A(a_1, a_2, \dots, a_n)$ with $0 < a_1 \le a_2 \le \cdots \le a_n$. Then the length of a shortest segment cut from the non-negative orthant K in $\mathbb{R}^n$ by a line through A is equal to $(a_1^{2/3} + a_2^{2/3})^{3/2}$.*

Proof. We may assume that $n \geq 3$. We will first prove that the length of a shortest segment cut from the set $Q = (\mathbb{R}_0^+)^2 \times \mathbb{R}^{n-2} \supset K$ by a line through A is given by the above formula, that such a segment is unique and that it lies in K. To do this, set $L_1 = \mathbb{R}_0^+ \times \mathbb{R}^{n-1}$ and $L_2 = \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}^{n-2}$. Let a line through A cut from Q a segment $[A_1A_2]$, where $A_1 \in L_1$ and $A_2 \in L_2$. Denote by Π the plane through A which is orthogonal to $\{0\} \times \{0\} \times \mathbb{R}^{n-2}$. Let A_1' and A_2' be the orthogonal projections of A_1 and A_2 on Π. Then $A_1' \in L_1$, $A_2' \in L_2$, $O \in [A_1'A_2']$, and $|A_1'A_2'| \leq |A_1A_2|$ with equality if and only if $A_1' = A_1$ and $A_2' = A_2$. Hence the problem is reduced to that for the right angle $Ol_1l_2 \subset K$, where $l_1 = L_1 \cap \Pi$ and $l_2 = L_2 \cap \Pi$. It remains to apply Corollary 7.

Let now $[BC]$ be a segment cut from K by a line through A. Setting $L_k = \mathbb{R}^{k-1} \times \mathbb{R}_0^+ \times \mathbb{R}_0^{n-k}$, $1 \leq k \leq n$, we may choose $i \neq j$ such that $B \in L_i$ and $C \in L_j$. Then the same reasoning as above shows that

$$| BC | \geq (a_i^{\frac{2}{3}} + a_j^{\frac{2}{3}})^{\frac{3}{2}} \geq (a_1^{\frac{2}{3}} + a_2^{\frac{2}{3}})^{\frac{3}{2}},$$

and the proposition is proved. □

This proof shows that for $n \geq 3$ the number of lines through A which cut from K segments of minimal length is equal to $k-1$ if $a_1 < a_2 = a_k < a_{k+1}$, and to $\binom{k}{2}$ if $a_1 = a_2 = a_k < a_{k+1}$. In particular, such a line is unique if and only if $a_2 < a_3$.

On the other hand, it is clear that among all lines $l \not\ni O$ through $A \notin K^o$ there is none which cuts from K a segment of minimal length.

Acknowledgements

The authors would like to thank the referee for carefully reading the manuscript and for bringing [4] to our attention.

References

[1] Bottema, O.: Verscheidenheden LXXIII. De uiterste waarden van een lijnstuk. Euclides **44**, 20–26 (1968-1969)

[2] Coxeter, H.S.M., van de Craats, J.: Philon lines in non-Euclidean planes. J. Geom. **48**, 26–55 (1993)

[3] Edmonds, A., Hajja, M., Martini, H.: Orthocentric simplices and their centers. Results Math. **47**, 266–295 (2005)

[4] Eves, H.: A Survey of Geometry, vol. 2. Allyn and Bacon, Boston (1965)

[5] Wetterling, W.W.E.: Philon's line generalized: an optimization problem from geometry. J. Optim. Theory Appl. **90**, 517–521 (1996)

Oleg Mushkarov and Nikolai Nikolov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev 8
1113 Sofia
Bulgaria
e-mail: muskarov@math.bas.bg

Nikolai Nikolov
Faculty of Information Sciences, State University of Library Studies and Information Technologies
Shipchenski prohod 69A
1574 Sofia
Bulgaria
e-mail: nik@math.bas.bg

Received: February 26, 2026.
Accepted: May 25, 2026.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.