Received: 16 January 2025 | Revised: 5 July 2025 | Accepted: 11 July 2025

DOI: 10.1002/mana.70020

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Twistor spaces, conformal changes, and metric connections on the base manifold

Johann Davidov | **Oleg Mushkarov**

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Correspondence
Johann Davidov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev St. Bl.8, 1113 Sofia, Bulgaria.
Email: jtd@math.bas.bg

Funding information
National Science Fund of Bulgaria, Grant/Award Number: KP-06-N82/6

Abstract
The following problems related to almost complex structures on a twistor space are discussed: (1) when two almost complex structures defined by a morphism of the twistor space and two conformal metrics on the base manifold coincide; (2) when the Atiyah–Hitchin–Singer or Eells–Salamon almost complex structures on the twistor space defined by two metric connections on the base Riemannian manifold coincide.

KEYWORDS
twistor spaces, almost complex structures

1 | INTRODUCTION

The twistor space $\mathcal{Z}$ of an even-dimensional Riemannian manifold (M, g) is a bundle over M that parameterizes the orthogonal complex structures on the tangent spaces of M. If M is oriented, one can consider the bundles over M parameterizing the orthogonal complex structures yielding the given or the opposite orientation of M. These bundles are respectively called positive and negative twistor space of (M, g), and if M is connected, their total spaces are the connected components of $\mathcal{Z}$.

The manifold $\mathcal{Z}$ admits two remarkable almost complex structures introduced respectively by Atiyah–Hitchin–Singer [2] and Eells–Salamon [10] in the case when the dimension of M is four. The twistor construction has been extended to any even-dimensional Riemannian manifold by Bérard–Bergery–Ochiai [4] and I. Skornyakov [15], see also O'Brian–Rawnsley [12]), and Salamon [14]. It has been observed by Deschamps [8] that one can define other almost complex structures on a twistor space by means of a morphism f of the twistor bundle. If $f = \mathrm{id}$, the corresponding almost complex structure coincides with that of Atiyah–Hitchin–Singer and if $f = -\mathrm{id}$ it is the conjugate to the Eeells–Salamon one. In fact, LeBrun [11] was the first to use morphisms of the twistor bundle to construct complex structures with interesting properties (see Example 1).

It is well-known (see e.g., [2, 12]) that the Atiyah–Hitchin–Singer almost complex structure is conformally invariant in the sense that it does not change under conformal changes of the base metric. This leads to the natural question when two almost complex structures $\mathcal{J}_f$ and $\widetilde{\mathcal{J}}_f$ on the twistor space defined by means of a morphism f of the twistor bundle and two conformal metrics $\widetilde{g} = e^{2\varphi}g$ on the base manifold coincide. We prove that if M is oriented and the function φ has no critical points, $\mathcal{J}_f = \widetilde{\mathcal{J}}_f$ on the positive twistor space if and only if $f = \mathrm{id}$, that is, $\mathcal{J}_f$ is the Atiyah–Hitchin–Singer almost complex structure (see Theorem 1).

The Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures on a twistor space are usually defined using the Levi-Civita connection of the base manifold. But such structures can be defined in a similar way using any metric connection [9]. Thus, another natural question is when two metric connections D and $\widetilde{D}$ on a Riemannian manifold (M, g) determine the same Atiyah–Hitchin–Singer or Eells–Salamon almost complex structures. We answer this question in terms of the irreducible components of the difference tensor $A(X,Y,Z) = g(\widetilde{D}_X Y - D_X Y, Z)$ under the action of the orthogonal group $O(n), n = \dim M$ on $TM \bigotimes \Lambda^2 TM$ [16] (see Theorem 2).

The paper is organized as follows. In Section 2, we recall some basic information about the twistor space of an even-dimensional Riemannian manifold and the definitions of the Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures. The so-called compatible almost complex structures (in the sense of Deschamps [8]) defined by means of morphisms of the twistor bundle are also considered. Then, in Sections 3 and 4, we prove Theorems 1 and 2, respectively, by analyzing the algebraic conditions on the difference tensor of the Levi-Civita connections of two conformal metrics and that of two arbitrary metric connections on the base manifold.

2 | BASICS OF TWISTOR SPACES

2.1 | The manifold of orthogonal linear complex structures

Let V be a real vector space of even dimension $n = 2m$ endowed with a Euclidean metric g. Denote by $F(V)$ the set of g-orthogonal complex structures on V. This set has the structure of an embedded smooth submanifold of the vector space $\mathfrak{so}(V)$ of skew-symmetric endomorphisms of (V, g). The tangent space of $F(V)$ at a point J consists of endomorphisms $Q \in \mathfrak{so}(V)$ anti-commuting with J, and we have the decomposition

$$\mathfrak{so}(V) = T_J F(V) \oplus \{S \in \mathfrak{so}(V) : \; SJ - JS = 0\}$$

which is orthogonal with respect to the metric $G(S,T) = -\frac{1}{2}\mathrm{Trace}(S \circ T)$ on $\mathfrak{so}(V)$. The smooth manifold $F(V)$ admits a natural almost-complex structure $\mathcal{J}$ defined by $\mathcal{J}Q = J \circ Q$ for $Q \in T_J F(V)$. This structure is compatible with the metric G. It is not hard to find an explicit formula for the Levi-Civita connection ∇ of G and to see that $\nabla \mathcal{J} = 0$, so $(G, \mathcal{J})$ is a Kähler structure on $F(V)$. In particular, $\mathcal{J}$ is an integrable structure, hence it is induced by complex coordinates. One can also show that the metric G is Einstein.

The group $O(V) \cong O(2m)$ of orthogonal transformations of V acts on $F(V)$ by conjugation and the isotropy subgroup at a fixed complex structure J_0 is isomorphic to the unitary group $U(m)$. Therefore, $F(V)$ can be identified with the homogeneous space $O(2m)/U(m)$. In particular, $\dim F(V) = m^2 - m$. Note also that the manifold $F(V)$ has two connected components $F_{\pm}(V)$: if we fix an orientation on V, these components consist of the complex structures on V compatible with the metric g and inducing the positive/negative orientation of V; each of them has the homogeneous representation $SO(2m)/U(m)$.

The metric g of V induces a metric on $\Lambda^2 V$ given by

$$g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2}[g(v_1, v_3)g(v_2, v_4) - g(v_1, v_4)g(v_2, v_3)]. \tag{1}$$

Then we have an isomorphism $\mathfrak{so}(V) \cong \Lambda^2 V$, which sends $S \in \mathfrak{so}(V)$ to the 2-vector $S^\wedge$ for which

$$2g(S^\wedge, u \wedge v) = g(Su, v), \quad u, v \in V. \tag{2}$$

This isomorphism is an isometry with respect to the metric $\frac{1}{2}G$ on $\mathfrak{so}(V)$ and the metric g on $\Lambda^2 V$: if K_a and K_b are the skew-symmetric endomorphisms of V corresponding to the 2-vectors a and b,

$$g(a, b) = \frac{1}{2}G(K_a, K_b). \tag{3}$$

Suppose that V is oriented and of dimension four. Then, the Hodge star operator defines an endomorphism $*$ of $\Lambda^2 V$ with $*^2 = \mathrm{Id}$. Hence, we have the orthogonal decomposition

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V,$$

where $\Lambda^2_{\pm}V$ are the subspaces of $\Lambda^2 V$ corresponding to the (± 1)-eigenvalues of the operator $*$. Let (e_1, e_2, e_3, e_4) be an oriented orthonormal basis of V. Set

$$s_1^{\pm} = e_1 \wedge e_2 \pm e_3 \wedge e_4, \quad s_2^{\pm} = e_1 \wedge e_3 \pm e_4 \wedge e_2, \quad s_3^{\pm} = e_1 \wedge e_4 \pm e_2 \wedge e_3. \tag{4}$$

Then $(s_1^{\pm}, s_2^{\pm}, s_3^{\pm})$ is an orthonormal basis of $\Lambda^2_{\pm}V$. The orientation of $\Lambda^2_{\pm}V$ determined by this basis does not depend on the choice of the basis (e_1, e_2, e_3, e_4) (see, e.g., [5]).

It is easy to see that the isomorphism $\varphi \to \varphi^{\wedge}$ identifies $F_{\pm}(V)$ with the unit sphere $S(\Lambda^2_{\pm}V)$ of the Euclidean vector space $(\Lambda^2_{\pm}V, g)$ (so, the factor $1/2$ in (1) is chosen in order to have spheres of radius 1). Under this isomorphism, if $J \in F_{\pm}(V)$, the tangent space $T_J F(V) = T_J F_{\pm}(V)$ is identified with the orthogonal complement $(\mathbb{R}J^{\wedge})^{\perp}$ of the space $\mathbb{R}J^{\wedge}$ in $\Lambda^2_{\pm}V$. Moreover, the restriction of the complex structure $\mathcal{J}$ to $F_{\pm}(V)$ is identified with the complex structure of the unit sphere in $\Lambda^2_{\pm}V$. Thus, $(\mathcal{J}Q)^{\wedge} = \pm(J^{\wedge} \times Q^{\wedge})$ where $\times$ is the vector-cross product on the three-dimensional Euclidean space $(\Lambda^2_{\pm}V, g)$ endowed with the orientation defined above.

2.2 | The twistor space of an even-dimensional Riemannian manifold

Let (M, g) be a Riemannian manifold of even dimension. The twistor space of (M, g) is the bundle $\pi : \mathcal{Z} \to M$ whose fiber at a point $p \in M$ is the manifold $F(T_pM)$ of all complex structures of the tangent space T_pM compatible with the metric g_p. If M is oriented, we can consider the bundles over M whose fiber at p is the manifold of compatible complex structures yielding the positive and, respectively, the negative orientation of T_pM. The latter bundles are disjoint open subsets of $\mathcal{Z}$ and are frequently called the positive and the negative twistor spaces of (M, g), respectively. They are usually denoted by $\mathcal{Z}_+$ and $\mathcal{Z}_-$. If M is connected and oriented, they are the connected components of the manifold $\mathcal{Z}$. Clearly, changing the orientation interchanges the role of the positive and negative twistor spaces. If M is oriented and of dimension four, we have the decomposition

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$$

under the action of the Hodge star operator, $\Lambda^2_{\pm}TM$ being the subbundles corresponding to the (± 1)-eigenvalues. The map $J \to J^{\wedge}$ identifies the positive (negative) twistor space with the unit sphere bundle $S(\Lambda^2_+TM)$ (resp. $S(\Lambda^2_-TM)$).

Let D be a metric connection on a Riemannian manifold (M, g). Then, one can define two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ introduced, respectively, by Atiyah–Hitchin–Singer [2] and Eells–Salamon [10] in the case when the base manifold M is of dimension four and D is the Levi-Civita connection of (M, g).

The bundle $\mathcal{Z}$ can be considered as a subbundle of the bundle $A(TM)$ of skew-symmetric endomorphisms of the tangent bundle TM. The metric connection D of (M, g) determines a connection on the bundle $A(TM)$ denoted also by D. The horizontal space of the vector bundle $A(TM)$ with respect to D at a point $J \in \mathcal{Z}$ is tangent to $\mathcal{Z}$. Hence, we have the splitting $T\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of $\mathcal{Z}$ into vertical and horizontal subbundles. For every $J \in \mathcal{Z}$, the horizontal subspace $\mathcal{H}_J$ of $T_J\mathcal{Z}$ is isomorphic to the tangent space $T_{\pi(J)}M$ via the differential π_{*J}, and we set

$$\mathcal{J}_k|\mathcal{H}_J = (\pi_*|\mathcal{H}_J)^{-1} \circ J \circ (\pi_*|\mathcal{H}_J), \quad k = 1, 2.$$

The vertical subspace $\mathcal{V}_J$ of $T_J\mathcal{Z}$ is the tangent space at J to the fiber of the bundle $\mathcal{Z}$ through J. This fiber is the complex manifold $F(T_{\pi(J)}M)$ we have discussed in Section 2.1 Denote its complex structure by $\mathcal{J}_{\mathcal{V}_J}$. Then the almost-complex structures $\mathcal{J}_k$ are defined on the vertical space $\mathcal{V}_J$ by

$$\mathcal{J}_1|\mathcal{V}_J = \mathcal{J}_{\mathcal{V}_J}, \quad \mathcal{J}_2|\mathcal{V}_J = -\mathcal{J}_{\mathcal{V}_J}.$$

The above construction of the almost complex structures $\mathcal{J}_k$ can be generalized in the following way [8]. Let $f : \mathcal{Z} \to \mathcal{Z}$ be a fiber-preserving smooth map of the twistor bundle. Then it defines an almost complex structure $\mathcal{J}_f$ on $\mathcal{Z}$ by

$$\mathcal{J}_f|\mathcal{H}_J = (\pi_*|\mathcal{H}_J)^{-1} \circ f(J) \circ (\pi_*|\mathcal{H}_J), \quad \mathcal{J}_f|\mathcal{V}_J = \mathcal{J}_{\mathcal{V}_J}$$

where $J \in \mathcal{Z}$. It is obvious that $\mathcal{J}_{\mathrm{id}} = \mathcal{J}_1$, the Atiyah–Hitchin–Singer almost structure, while $\mathcal{J}_{-\mathrm{id}} = -\mathcal{J}_2$, the conjugate almost complex structure of the Eells–Salamon one.

Here are some examples of compatible almost complex structures.

Example 1. As we noted in Section 1, LeBrun [11] was the first to use self-maps of a twistor space to construct complex structures with interesting properties. He considered the twistor space $\mathcal{Z}_+$ of a $K3$-surface M with a hyper-Kähler metric, in which case $\mathcal{Z}_+$ is diffeomorphic to $M \times \mathbb{CP}^1$. LeBrun used the complex structures on $\mathcal{Z}_+$ defined by the maps $f_m : M \times \mathbb{CP}^1 \to M \times \mathbb{CP}^1$ given by $f_m(x, [u, v]) = (x, [u^{m-1}, v^{m-1}])$, $x \in M$ for $m = 1, 2, \ldots$ to show that the Chern numbers c_1^3 and $c_1 c_2$ of a compact complex 3-manifold are not determined by the topology of the underlying smooth manifold.

Example 2. Let (M, g, K) be an almost Hermitian manifold. Then one can define a morphism f_K of $\mathcal{Z}$ by

$$f_K(J) = K_{\pi(J)}, \quad J \in \mathcal{Z}.$$

If we endow M with the orientation determined by the almost complex structure K, f_K is a morphism of the positive twistor space $\mathcal{Z}_+$. We refer the reader to [1, 7, 8] where geometric properties of the compatible almost complex structure $\mathcal{J}_{f_K}$ have been studied.

Example 3. Let (M, g, K) be an almost Hermitian manifold of dimension four. Consider M with orientation induced by the almost complex structure K. In this case, we identify the positive twistor space $\mathcal{Z}_+$ with the unit sphere bundle $S(\Lambda^2_+ TM)$.

Let ω be the unit 2-vector field corresponding to the almost Hermitian structure (g, K), that is,

$$g(\omega, X \wedge Y) = \frac{1}{2} g(KX, Y), \quad X, Y \in TM.$$

Then, as in [8], we can define morphisms of $\mathcal{Z}_+$ by using the stereographic projection χ_σ of every fiber $(\mathcal{Z}_+)_{\pi(\sigma)}$ from the point $\omega_{\pi(\sigma)}$ onto the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$, the orthogonal complement being taken in $\Lambda^2_+ T_{\pi(\sigma)} M$. As usual, we also set $\chi_\sigma(\omega_{\pi(\sigma)}) = \infty$, the "ideal" element of the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$. Let $\lambda = a + ib \in \mathbb{C}$ and set $F_\lambda(\zeta) = \lambda\zeta$ for $\zeta \in (\mathbb{R}\omega_{\pi(\sigma)})^\perp$. Then

$$f_\lambda^+(\sigma) = \chi_\sigma^{-1} \circ F_\lambda \circ \chi_\sigma(\sigma)$$

is a self-map of $\mathcal{Z}_+$ whose restriction to any fiber is holomorphic. Similarly, denote by ψ_σ the stereographic projection of $(\mathcal{Z}_+)_{\pi(\sigma)}$ from the point $-\omega_{\pi(\sigma)}$ onto the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$ and set

$$f_\lambda^-(\sigma) = \chi_\sigma^{-1} \circ F_\lambda \circ \psi_\sigma(\sigma).$$

In this way, we obtain another self-map of $\mathcal{Z}_+$ whose restriction to any fiber is anti-holomorphic. Clearly, the points $f_\lambda^-(\sigma)$ and $f_\lambda^+(\sigma)$ are symmetric with respect to the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$.

It is not hard to see that the maps $f_\lambda^\pm : \mathcal{Z}_+ \to \mathcal{Z}_+$ are given by the following explicit formula [1]:

$$\begin{aligned} f_\lambda^\pm(\sigma) = {} & [(a^2 + b^2 + 1) + (a^2 + b^2 - 1) g(\sigma, \omega_{\pi(\sigma)})]^{-1} \\ & \{2a\sigma - 2b\sigma \times \omega_{\pi(\sigma)} - 2ag(\sigma, \omega_{\pi(\sigma)})\omega_{\pi(\sigma)} \\ & \pm [(a^2 + b^2 - 1) + (a^2 + b^2 + 1) g(\sigma, \omega_{\pi(\sigma)})]\omega_{\pi(\sigma)}\}. \end{aligned} \tag{5}$$

Denote by $\mathcal{J}_\lambda^\pm$ the almost complex structure on $\mathcal{Z}_+$ defined by means of the map $f_\lambda^\pm$. Then $\mathcal{J}_1^+$ is the Atiyah–Hitchin–Singer almost complex structure and $\mathcal{J}_1^-$ is the conjugate of the Eells–Salamon almost complex structure. Also, $\mathcal{J}_0^\pm$ is the almost complex structure on $\mathcal{Z}_+$ yielded by the almost complex structure $\mp K$ on M and discussed in Example 1.

Example 4. Let (M, g) be an even-dimensional Riemannian manifold and $\mathcal{Z}$ its twistor space. For any non-vanishing vector field E on M, we define a morphism f_E of $\mathcal{Z}$ in the following way: $f_E(J) = -J$ on $Span(E, JE)$ and $f_E(J) = J$ on the orthogonal complement of $Span(E, JE)$. Note that $f_E(J)$ and J determine opposite orientations on $T_{\pi(J)}M$. Thus, if the manifold M is oriented, then the morphism $f_E : \mathcal{Z} \to \mathcal{Z}$ interchanges the positive and the negative twistor spaces $\mathcal{Z}_+$ and $\mathcal{Z}_-$.

We refer the reader to [1, 7, 8, 13] where the geometric and analytic properties of the compatible almost complex structures $\mathcal{J}_f$ have been studied.

Notation. In what follows, the positive twistor space of an oriented Riemannian manifold will be denoted by $\mathcal{Z}$.

3 | CONFORMAL INVARIANCE OF COMPATIBLE ALMOST COMPLEX STRUCTURES ON TWISTOR SPACES

In this section, we will show that the Atiyah–Hitchin–Singer almost complex structure is the only compatible almost complex structure on the positive (negative) twistor space of an oriented even-dimensional Riemannian manifold which is invariant under conformal changes of the base metric.

Let g and $\widetilde{g}$ be two conformal metrics on a smooth manifold M. Then clearly every g-orthogonal endomorphism of any tangent space of M is $\widetilde{g}$-orthogonal and vice versa. Thus, the Riemannian manifolds (M, g) and $(M, \widetilde{g})$ have the same twistor space.

Theorem 1. *Let $f : \mathcal{Z} \to \mathcal{Z}$ be a fiber-preserving smooth map of the positive twistor space $\mathcal{Z}$ of an oriented even-dimensional Riemannian manifold (M, g) and let $\widetilde{g} = e^{2\varphi} g$ be a conformal metric. Suppose that the function φ has no critical points. Then the almost complex structures $\mathcal{J}_f$ and $\widetilde{\mathcal{J}_f}$ on $\mathcal{Z}$ corresponding to the Levi-Civita connections of g and $\widetilde{g}$ coincide if and only if $f = id$.*

Proof. Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of g and $\widetilde{g}$, respectively. They are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\varphi)Y + Y(\varphi)X - g(X, Y)\nabla\varphi \tag{6}$$

where $\nabla\varphi$ is the gradient of φ defined by $g(\nabla\varphi, Z) = Z(\varphi)$.

Let $\pi : A(TM) \to M$ be the projection map of the bundle $A(TM), J \in \mathcal{Z}$ and $X \in T_{\pi(J)}M$. The horizontal lifts of X to $T_J\mathcal{Z}$ with respect to ∇ and $\widetilde{\nabla}$ will be denoted by X^h_J and $\widetilde{X}^h_J$. Let S be a section of $\pi : \mathcal{Z} \to M$ such that $S(p) = J$. Then

$$X^h_J + \nabla_X S = S_*(X) = \widetilde{X}^h_J + \widetilde{\nabla}_X S. \tag{7}$$

On the other hand, using identity (6), we get

$$(\widetilde{\nabla}_X S)(Y) = (\nabla_X S)(Y) + V_{J,X}(Y),$$

where $V_{J,X} : T_pM \to T_pM$ is the linear map defined by

$$V_{J,X}(Y) = (JY)(\varphi)X - g(X, JY)\nabla\varphi - Y(\varphi)JX + g(X, Y)J\nabla\varphi. \tag{8}$$

The endomorphism $V_{J,X}$ of T_pM is skew-symmetric with respect to g (as well as with respect to $\widetilde{g}$) and anti-commutes with J. Thus, $V_{J,X}$ is a vertical tangent vector to $\mathcal{Z}$ at the point J and the identity (7) can be written as

$$X^h_J = \widetilde{X}^h_J + V_{J,X}. \tag{9}$$

By the definitions of $\mathcal{J}_f$ and $\widetilde{\mathcal{J}}_f$, we get from (9)

$$\begin{aligned}\widetilde{\mathcal{J}}_f X^h_J &= \widetilde{(f(J)X)}^h_J + JV_{J,X} = (f(J)X)^h_J - V_{J,f(J)X} + JV_{J,X}\\ &= \mathcal{J}_f X^h_J - V_{J,f(J)X} + JV_{J,X}.\end{aligned}$$

Having in mind that $\mathcal{J}_f|\mathcal{V}_J = \widetilde{\mathcal{J}}_f|\mathcal{V}_J$, we conclude that $\widetilde{\mathcal{J}}_f = \mathcal{J}_f$ if and only if

$$V_{J,f(J)X} = JV_{J,X} \tag{10}$$

for every $J \in \mathcal{Z}$ and every $X \in T_{\pi(J)}M$.

To prove the theorem, it remains to show that (10) implies $f = \mathrm{id}$. To do this, we observe first that $JV_{J,X} = V_{J,JX}$ and rewrite (10) as

$$V_{J,(f(J)-J)X} = 0 \tag{11}$$

for all $J \in \mathcal{Z}$ and $X \in T_{\pi(J)}M$. Note also that the vector field $E = \nabla\varphi$ does not vanish, and using (8) one can easily prove that $V_{J,W} = 0$ if and only if $W \in \mathrm{Span}(E, JE)$. Hence, $(f(J) - J)X \in \mathrm{Span}(E, JE)$ for all $X \in T_{\pi(J)}M$. In particular, $(f(J) - J)E \in \mathrm{Span}(E, JE)$, and the fact that $g((f(J) - J)E, E) = 0$ implies that $f(J)(E) = \pm JE$. Then

$$f(J)(JE) = \pm(f(J))^2(E) = \pm(J(JE)$$

and therefore $f(J) = \pm J$ on $\mathrm{Span}(E, JE)$. Let $X \perp \mathrm{Span}(E, JE)$ be an arbitrary tangent vector. Then $JX \perp \mathrm{Span}(E, JE)$ and using the identity $f(J)(X) = JX + W$, where $W \in \mathrm{Span}(E, JE)$, we get

$$\| W \|^2 = g(f(J)(X), W) = -g(X, f(J)(W)) = \mp g(X, JW) = 0.$$

Thus, $W = 0$ and we conclude that $f(J) = J$ on the orthogonal complement of $\mathrm{Span}(E, JE)$. Now, the fact that $f(J)$ and J determine the same orientation shows that the case $f(J) = -J$ on $\mathrm{Span}(E, JE)$ is impossible. Hence, $f(J) = J$ for all $J \in \mathcal{Z}$. □

Remarks.

1. If $\nabla\varphi(p) = 0$ at a point $p \in M$, then $V_{J,X} = 0$ for every $J \in \mathcal{Z}_p$, the fiber of $\mathcal{Z}$ at p, and every $X \in T_{\pi(J)}M$. Hence, $\mathcal{J}_f = \widetilde{\mathcal{J}}_f$ on $T_J\mathcal{Z}$ for every $J \in \mathcal{Z}_p$ if p is a critical point of φ. This and Theorem 1 imply the well-known result that the Atiyah–Hitchin–Singer almost complex structures $\mathcal{J}_1$ and $\widetilde{\mathcal{J}}_1$ on the twistor space of two conformal metrics g and $\widetilde{g}$ on a manifold coincide. Conversely, by a result in [3], if the twistor spaces of two Riemannian metrics g and $\widetilde{g}$ on the same manifold are biholomorphic with respect to the Atiyah–Hitchin–Singer almost complex structures by a fiber-preserving map, then the metrics g and $\widetilde{g}$ are conformally equivalent. The case of twistor spaces of two Riemannian manifolds biholomorphic with respect to the Atiyah–Hitchin–Singer or Eells–Salamon almost complex structures is discussed in [6].
2. Consider the whole twistor space of an arbitrary even-dimensional Riemannian manifold (M, g) and a conformal to g metric $\widetilde{g} = e^{2\varphi}g$ with $\nabla\varphi \neq 0$ at every point. Then the above proof shows that the morphisms f of the twistor space such that the almost complex structures $\mathcal{J}_f$ and $\widetilde{\mathcal{J}}_f$ coincide are $f = id$ and $f = f_{\nabla\varphi}$, where $f_{\nabla\varphi}$ is the morphism of $\mathcal{Z}$ defined by means of the vector field $\nabla\varphi$ as in Example 1.

4 | ALMOST COMPLEX STRUCTURES ON TWISTOR SPACES DEFINED BY METRIC CONNECTIONS

Let (M, g) be an oriented Riemannian manifold, D and $\widetilde{D}$ two metric connections on it, and $A = \widetilde{D} - D$, where

$$A(X, Y) = \widetilde{D}_X Y - D_X Y \ , \ X, Y \in TM.$$

Since the connections are both metric,

$$g(A(X, Y), Z) = -g(A(X, Z), Y).$$

Set $A(X,Y,Z) = g(A(X,Y),Z)$ and

$$c(A)(Z) = \frac{1}{n-1}\sum_{i=1}^{n} A(E_i, E_i, Z)$$

for an arbitrary orthonormal frame $(E_1, \dots, E_n)$ of TM. Then $A \in TM \bigotimes \Lambda^2 TM$ and by [16, Theorems 3.1 and 3.2], the tensor A has three irreducible components A_1, A_2, A_3 under the action of the orthogonal group $O(n)$ on $TM \bigotimes \Lambda^2 TM$. They are given by

$$A_1(X,Y,Z) = g(X,Y)c(A)(Z) - g(X,Z)c(A)(Y)\,, \tag{12}$$

$$A_2(X,Y,Z) = \frac{1}{3}\underset{X,Y,Z}{\mathfrak{S}} A(X,Y,Z) \tag{13}$$

and $A_3 = A - A_1 - A_2$. Note that A_2 is totaly skew-symmetric and

$$\underset{X,Y,Z}{\mathfrak{S}} A_3(X,Y,Z) = 0\,,\ c(A_3) = 0 \tag{14}$$

for all $X, Y, Z \in TM$.

Theorem 2. *Let D and $\widetilde{D}$ be two metric connections on an oriented even-dimensional Riemannian manifold (M,g) with positive twistor space $\mathcal{Z}$ and let $A = \widetilde{D} - D$.*

(i) D and $\widetilde{D}$ yield the same Atiyah–Hitchin–Singer almost complex structure on $\mathcal{Z}$ if and only if
  (a) $dimM = 4$ and the 2-form $\iota_X A_3$ is self-dual for every $X \in TM$
  or
  (b) $dimM \geq 6$ and $A = A_1$.

(ii) D and $\widetilde{D}$ yield the same Eells–Salamon almost complex structure on $\mathcal{Z}$ if and only if
  (a) $dimM = 4$ and the 2-form $\iota_X A$ is self-dual for every $X \in TM$
  or
  (b) $dimM \geq 6$ and $D = \widetilde{D}$.

Proof. Recall that D and $\widetilde{D}$ yield the same Atiyah–Hitchin–Singer almost complex structure on $\mathcal{Z}_+$ if and only if

$$A(X,Y,Z) - A(JX,JY,Z) - A(JX,Y,JZ) - A(X,JY,JZ) = 0 \tag{15}$$

for every $J \in \mathcal{Z}$ [9, Lemma 2.4]. This follows from the observation that if X^h_J and $\widetilde{X}^h_J$ are the horizontal lifts of a tangent vector $X \in T_{\pi(J)}M$ with respect to the connections D and $\widetilde{D}$, respectively, then $X^h_J = \widetilde{X}^h_J + V_{J,X}$ where $V_{J,X}$ is the vertical tangent vector to $\mathcal{Z}$ at J given by

$$V_{J,X}(Y) = A(X,JY) - JA(X,Y), \quad Y \in T_{\pi(J)}M.$$

It also follows that D and $\widetilde{D}$ determine the same Eells–Salamon almost complex structure exactly when

$$A(X,Y,Z) + A(JX,JY,Z) + A(JX,Y,JZ) - A(X,JY,JZ) = 0. \tag{16}$$

for every $J \in \mathcal{Z}$.

Define a tensor B on M by

$$g(B(Y,Z),X) = A(X,Y,Z). \tag{17}$$

Clearly, B is skew-symmetric, and we extend it to a linear map $B : \Lambda^2 TM \to TM$. In terms of B, identities (15) and (16) can be written as

$$g(B(Y \wedge Z - JY \wedge JZ), X) + \varepsilon g(B(JY \wedge Z + Y \wedge JZ), JX) = 0, \tag{18}$$

with $\varepsilon = -1$ for the Atiyah–Hitchin–Singer almost complex structure and $\varepsilon = +1$ for the Eells–Salamon one.

Let $\dim M = 2m$, and take an orthonormal system of vector fields $e = \{E_1, \dots, E_4\}$ in a neighborhood of a point $p \in M$. If $\dim M \geq 6$, we can complete this system to an oriented orthonormal local frame $(E_1, \dots, E_4, E_5, \dots, E_{2m})$. If $\dim M = 4$, we will assume that the orthonormal local frame $(E_1, \dots, E_4)$ is oriented. It is convenient to set

$$s_1 = E_1 \wedge E_2 + E_3 \wedge E_4, \quad s_2 = E_1 \wedge E_3 + E_4 \wedge E_2, \quad s_3 = E_1 \wedge E_4 + E_2 \wedge E_3. \tag{19}$$

Let J_e be the almost complex structure defined in a neighborhood of the point p by $E_2 = J_e E_1, \dots, E_{2m} = J_e E_{2m-1}$. Then J_e is a section of $\mathcal{Z}$, and identity (18) is satisfied for every $Y, Z \in \{E_1, E_2, E_3, E_4\}$ exactly when

$$g(B(s_2), X) + \varepsilon g(B(s_3), J_e X) = 0 \tag{20}$$

for every $X \in TM$. Let S be the $SO(4)$-matrix

$$S = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

where a, b, c, d are real numbers with $a^2 + b^2 + c^2 + d^2 = 1$. Denote by $e' = \{E_1', \dots, E_4'\}$ the orthonormal system of vector fields obtained by the left action of the matrix S to the column $(E_1, \dots, E_4)^t$. Let s_1', s_2', s_3' be the 2-vectors defined by means of $E_1', \dots, E_4'$ via (19). Then

$$\begin{aligned} s_1' &= (a^2 + b^2 - c^2 - d^2)s_1 - 2(ad - bc)s_2 + 2(ac + bd)s_3, \\ s_2' &= 2(ad + bc)s_1 + (a^2 + c^2 - b^2 - d^2)s_2 - 2(ab - cd)s_3, \\ s_3' &= -2(ac - bd)s_1 + 2(ab + cd)s_2 + (a^2 + d^2 - b^2 - c^2)s_3. \end{aligned}$$

Applying (20) for s_2, s_3 replaced with s_2', s_3' and for $X = E_1'$, $J_{e'}X = E_2'$, we get a polynomial identity in a, b, c, d with coefficients linear combinations of $g(B(s_i), E_j)$. This identity and (20) imply

$$\begin{gathered} g(B(s_1), E_1) = g(B(s_2), E_4) = \varepsilon g(B(s_3), E_3), \\ g(B(s_1), E_2) = -\varepsilon g(B(s_2), E_3) = g(B(s_3), E_4), \\ (1 - \varepsilon)g(B(s_1), E_3) = 2\varepsilon g(B(s_2), E_2) = 2g(B(s_3), E_1), \\ g(B(s_1), E_4) = -g(B(s_2), E_1) = \varepsilon g(B(s_3), E_2), \\ (1 + \varepsilon)g(B(s_1, E_1) = (1 + \varepsilon)g(B(s_1), E_2) = 0, \\ [2 + \tfrac{1}{2}\varepsilon(1 - \varepsilon)]g(B(s_1), E_3) = -g(B(s_2), E_2), \\ (1 - \varepsilon)g(B(s_1, E_4) = 2g(B(s_3), E_2). \end{gathered} \tag{21}$$

Proof of (ii). In this case, $\varepsilon = 1$ and we get from (21) that

$$g(B(s_i), E_j) = 0, \quad i = 1, 2, 3, j = 1, \dots, 4. \tag{22}$$

If $\dim M = 4$, then $B(\sigma) = 0$ for every $\sigma \in \Lambda^2_+ TM$ and by (17) the 2-form $\iota_X A$ is self-dual for every $X \in TM$. If $\dim M \geq 6$, we can apply identities (22) for the orthogonal system $\{-E_1, E_2, E_3, E_4\}$. The identities obtained and (22) imply

$$g(B(E_i, E_j), E_k) = 0, \quad i, j, k = 1, \dots, 4.$$

These identities hold for any orthonormal system of four vector fields, hence $B = 0$. Thus, $A = 0$, that is, the connections D and $\widetilde{D}$ coincide

The converse statement is trivial in the case $\dim M = 6$. It follows in the case $\dim M = 4$ from the observation that if the 2-form $\iota_X A$ is self-dual for every $X \in TM$, then identity (18) is satisfied since the 2-vectors $Y \wedge Z - JY \wedge JZ$ and $JY \wedge Z + Y \wedge JZ$ belong to $\Lambda^2_+ TM$ for every $J \in \mathcal{Z}$ and every $Y, Z \in T_{\pi(J)} M$.

Proof of (i). In this case $\varepsilon = -1$ and identities (21) give

$$\begin{aligned} g(B(s_1), E_1) &= g(B(s_2), E_4) = -g(B(s_3), E_3), \\ g(B(s_1), E_2) &= g(B(s_2), E_3) = g(B(s_3), E_4) \\ g(B(s_1), E_3) &= -g(B(s_2), E_2) = g(B(s_3), E_1), \\ g(B(s_1), E_4) &= -g(B(s_2), E_1) = -g(B(s_3), E_2). \end{aligned} \tag{23}$$

A straightforward and easy computation shows that the tensor A_1 satisfies condition (15) and the tensor A_2 does so if $\dim M = 4$ (without any assumption on A). Therefore, if $\dim M = 4$, the tensor A satisfies identity (15) if and only if A_3 does.

Set

$$g(B_i(Y, Z), X) = g(A_i(X, Y), Z), \quad i = 1, 2, 3.$$

Suppose first that $\dim M = 4$. Since A_3 satisfies condition (15), the tensor B_3 satisfies identities (23). Moreover,

$$\underset{X,Y,Z}{\mathfrak{S}} g(B_3(X, Y), Z) = 0$$

and

$$\sum_{i=1}^{4} g(B_3(E_i, Z), E_i) = 0.$$

Then

$$g(B_3(s_1), E_1) + g(B_2(s_2), E_4) - g(B_3(s_3), E_3) = 0$$

and the first identity of (23) implies that

$$g(B_3(s_1), E_1) = g(B_3(s_2), E_4) = -g(B_3(s_3), E_3) = 0.$$

Similarly, all inner products involving B_3 in (23) vanish and it follows that $B_3(\sigma) = 0$ for every $\sigma \in \Lambda^2_+ TM$. Conversely, if the latter identity holds, identity (18) is satisfied since, as we noted above, the 2-vectors $Y \wedge Z - JY \wedge JZ$ and $JY \wedge Z + Y \wedge JZ$ belong to $\Lambda^2_+ TM$. Thus, if $\dim M = 4$ the connections D and $\widetilde{D}$ yield the same Atiyah–Hitchin–Singer almost complex structure on $\mathcal{Z}$ if and only if the 2-form $\iota_X A_3$ is self-dual for every $X \in TM$.

Suppose now $\dim M \geq 6$. Then, we can apply identities (23) for the orthogonal system $\{-E_1, E_2, E_3, E_4\}$ and compare the identities obtained with those in (23). Setting $B_{ijk} = g(B(E_i \wedge E_j), E_k)$, we see that

$$B_{212} = B_{313} = B_{414}, \quad B_{121} = B_{323} = B_{424}, \quad B_{131} = B_{232} = B_{434},$$
$$B_{141} = B_{242} = B_{343}, \quad B_{123} = B_{231} = B_{312}, \quad B_{124} = B_{241} = B_{412},$$
$$B_{134} = B_{341} = B_{413}, \quad B_{234} = B_{342} = B_{423}.$$

Since these identities hold for any orthonormal system of four vector fields, it follows that for every orthonormal frame $(E_1, \dots, E_{2m})$, we have $B_{ijk} = B_{jki} = B_{kij}$ for any three different indexes. In this case, $B_{ijk} = B_{kij} = -B_{ikj}$, so B_{ijk} is skew-symmetric for different indexes i, j, k. Moreover, for any fixed i, $B_{jij} = B_{kik}$ if $j \neq i, k \neq i$, $j, k = 1, \dots, 2m$. Hence we can define smooth functions f_i, $i = 1, \dots, 2m$, such that $f_i = B_{jij}$ for $j = 1, \dots, 2m$. For $X, Y, Z \in TM$, set $X_i = g(X, E_i)$, and similarly for Y_i and Z_i. Then

$$\begin{aligned} g(B(Y,Z),X) &= \sum_{i,j,k=1}^{2m} Y_i Z_j X_k B_{ijk} \\ &= \sum_{i,j=1}^{2m} Y_i Z_j X_i B_{iji} + \sum_{i,k=1}^{2m} Y_i Z_i X_k B_{ikk} + \sum_{i\neq j, j\neq k, k\neq i} Y_i Z_j X_k B_{ijk} \\ &= \sum_{i,j=1}^{2m} Y_i Z_j X_i B_{iji} - \sum_{i,k=1}^{2m} Y_i Z_i X_k B_{kik} + \sum_{i\neq j, j\neq k, k\neq i} Y_i Z_j X_k B_{ijk} \\ &= g(Y,X) \sum_{j=1}^{2m} g(Z, E_j) f_j - g(X,Z) \sum_{i=1}^{2m} g(Y_i, E_i) f_i + \sum_{i\neq j, j\neq k, k\neq i} Y_i Z_j X_k B_{ijk} \end{aligned}$$

Thus, setting

$$W = \sum_l f_l E_l \, , \; C(Y,Z,X) = \sum_{i\neq j, j\neq k, k\neq i} Y_i Z_j X_k B_{ijk},$$

we get

$$g(B(Y,Z),X) = g(Y,X)g(Z,W) - g(X,Z)g(Y,W) + C(Y,Z,X)$$

This shows that if $\dim M \geq 6$ and A satisfies the identities (15), in a neighborhood of every point of M, there exist a vector field W and a skew-symmetric 3-form C such that

$$A(X,Y,Z) = g(X,Y)g(Z,W) - g(X,Z)g(Y,W) + C(X,Y,Z).$$

The latter identity implies

$$g(W,Z) = \frac{1}{n-1} \sum_{i=1}^{n} g(A(E_i, E_i), Z) = c(A)(Z)$$

and therefore $A = A_1 + C$. The tensor A_1 satisfies identity (15), hence so does $C(X,Y,Z)$. Take any oriented orthonormal basis $e = (E_1, \dots, E_{2m})$ of a tangent space T_pM. Let J_e be the complex structure of T_pM for which $J_e E_{2k-1} = E_{2k}$, $k = 1, \dots, m$. Then

$$C(E_1, E_2, X) + C(E_3, E_4, X) - C(E_1, E_4, J_e X) - C(E_2, E_3, J_e X) = 0$$

for every $X \in T_pM$. In particular, setting $C_{ijk} = C(E_i, E_j, E_k)$,

$$C_{125} + C_{345} - C_{146} - C_{236} = 0.$$

Applying the latter identity for the basis

$$e' = (E_2, E_1, E_4, E_3, E_5, E_6, \dots, E_{2m}),$$

we get

$$C_{215} + C_{435} - C_{236} - C_{146} = 0.$$

The last two identities imply

$$C_{125} + C_{345} = 0.$$

Applying this identity for the basis

$$\mathrm{e}'' = (-E_1, E_2, E_3, E_4, E_5, -E_6, E_7, \dots, E_{2m})$$

we see that $C_{125} = 0$. It follows $C_{ijk} = 0$ for any three different indexes, that is, $C = 0$, hence $A = A_1$. Conversely, as we already noted A_1 satisfies identity (15) and (i) is proved completely. □

ACKNOWLEDGMENTS

The authors would like to thank the reviewer for pointing out the result in [3, Proposition 2.3.7]. The first named author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under contract KP-06-N82/6.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

No datasets were generated or analyzed during the current study.

ORCID

Johann Davidov https://orcid.org/0000-0001-9466-6011

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How to cite this article: J. Davidov and O. Mushkarov, *Twistor spaces, conformal changes, and metric connections on the base manifold*, Math. Nachr. **298** (2025), 3049–3060. https://doi.org/10.1002/mana.70020