MODELS FOR INVARIANT METRICS NEAR PSEUDOCONCAVE POINTS

NIKOLAI NIKOLOV AND PASCAL J. THOMAS

Abstract. We give precise estimates of some holomorphically invariant infinitesimal metrics near a pseudoconcave points in a wide family of "model" domains for that situation in $\mathbb{C}^2$. This extends to metrics (rather distances) the authors' previous results from [6, Section 4], and also takes into account defining functions more general than just power functions.

1. Introduction and results

1.1. **Motivations.** Holomorphically invariant metrics and distances have long proved useful for various questions in complex analysis and geometry, notably the mapping problem between domains of $\mathbb{C}^d$. It turns out that the presence of non-pseudoconvex points on the boundary of the domain gives rise to quantitatively different behavior of the invariant metrics in a neighborhood of those points; this was explored in [5, 3, 2, 1] among others. The interested reader should also consult the survey [4].

Typically, failure of pseudoconvexity at a boundary point p of a domain Ω is exploited by showing that some pseudoconcave model domain can be found with $G \subset \Omega$ and $p \in (\partial G) \cap (\partial \Omega)$. The behavior of holomorphic invariants in a range of such domains was studied in [6, Section 4]. Here we want to concentrate on infinitesimal metrics rather than distances, and generalize this to a wider range of models. Note however that we could have considered more general models (with a defining function also depending on $\operatorname{Im} z_1$ for instance, see [6]) and extended those estimates to higher dimensions.

1.2. **Basic definitions.** Let $\mathbb{D}$ stand for the unit disc in the complex plane. For an open set $\Omega \subset \mathbb{C}^d$, $\mathcal{O}(\mathbb{D}, \Omega)$ is the set of holomorphic maps from $\mathbb{D}$ to Ω and the Kobayashi-Royden metric is defined as follows:

$$\kappa_\Omega(z; X) := \inf\{\lambda^{-1} : \lambda > 0, \exists \varphi \in \mathcal{O}(\mathbb{D}, \Omega), \varphi(0) = z, \varphi'(0) = \lambda X\}.$$

This is a Finsler metric (that is, $\kappa_\Omega(z; \alpha X) = |\alpha| \kappa_\Omega(z; X)$, $\alpha \in \mathbb{C}$), contracting under holomorphic maps, and therefore invariant under biholomorphisms. It does not always satisfy the triangle inequality, so it makes sense to define, for $m \in \mathbb{N}$, $m \geq 1$, the m-th Kobayashi-Royden metrics

$$\kappa_\Omega^{(m)}(z; X) := \inf\{\sum_{j=1}^m \kappa_\Omega(z; X_j) : X_j \in \mathbb{C}^d, 1 \leq j \leq m, \text{ and } \sum_{j=1}^m X_j = X\}.$$

Clearly, $\kappa_\Omega^{(m)} \geq \kappa_\Omega^{(m+1)}$; $\hat{\kappa}_\Omega := \lim_{m\to\infty} \kappa_\Omega^{(m)}$ is the Kobayashi-Buseman metric, the largest Finsler metric less than the Kobayashi-Royden metric which satisfies the triangle inequality.

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As in [1], we define a metric that sits between the Kobayashi-Royden and Kobayashi-Buseman metrics. Recall that the *indicatrix* of a metric M_D at a base point z is

$$I_z M_D := \left\{ v \in T_z^{\mathbb{C}} D : M_D(z,v) < 1 \right\}.$$

The indicatrix of a metric which satisfies the triangle inequality is convex. The larger the indicatrices, the smaller the metric. We define $\widetilde{\kappa}_D$ to be *the largest invariant metric with pseudoconvex indicatrices,* i.e., since κ_D is the largest invariant metric, $I_z\widetilde{\kappa}_D$ is the envelope of holomorphy of $I_z\kappa_D$ for any $z \in D$.

Using plurisubharmonic functions, we can define the Sibony metric of a domain D: for $p \in D$, $X \in \mathbb{C}^d$,

$$S_D(p,X) := \sup\left\{ \partial\bar\partial u(p)(X,\bar X)^{1/2} := \left(\sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial z_i \partial \bar z_j}(p) X_i \bar X_j \right)^{1/2} : u \in A(p,D) \right\},$$

where $\mathbb{D}$ denotes a unit disc in $\mathbb{C}$, and $A(p,D)$ is the set of (plurisubharmonic) functions on D such that $u(p)=0$, u is $\mathcal{C}^2$ near p, $\log u$ is plurisubharmonic on D, and $0 \le u \le 1$ on D. The Sibony metric satisfies the triangle inequality, see for instance [2, Lemma 2].

Recall that for a domain $\Omega \subset \mathbb{C}^d$,

$$S_\Omega \le \hat\kappa_\Omega \le \widetilde{\kappa}_\Omega \le \kappa_\Omega. \tag{1}$$

1.3. **Results.** For ψ a continuous function from $[0,1]$ to $[0,\infty)$ with $\psi(0)=0$, $\psi(1)>0$ let us define the domain

$$G_\psi := \{z \in \mathbb{D}^2 : \operatorname{Re} z_1 < \psi(|z_2|)\}. \tag{2}$$

Note that this is always connected because $\psi(x) \ge 0$, and open because ψ is continuous.

Let $(x_N, x_T) \in \mathbb{C}^2$ and for $\delta \in (0;1)$, $p_\delta := (-\delta, 0) \in G_\psi$.

Standing assumption. We will restrict consideration to $\delta \in (0, \delta_0]$, with $\delta_0 < 1$. In all the estimates below, the constants which appear depend on ψ and δ_0.

Theorem 1. *If $\frac{\psi(x)}{x}$ is an increasing function on $(0;1)$, there exist $0 < c < C$ such that*

$$c\left(\frac{\psi^{-1}(\delta)}{\delta}|x_N| + |x_T| \right) \le S_{G_\psi}(p_\delta;(x_N,x_T)) \le M_{G_\psi}(p_\delta;(x_N,x_T)) \le C\left(\frac{\psi^{-1}(\delta)}{\delta}|x_N| + |x_T| \right),$$

where M_{G_ψ} stands for either $\kappa^{(2)}_{G_\psi}$ or $\widetilde{\kappa}_{G_\psi}$.

Remarks 1.

(a) When $\psi(x) = x^2$, the upper bounds for $\widetilde{\kappa}_{G_\psi}$ were already obtained in [1, Lemma 7], as well as the lower bound for S_{G_ψ} when $\psi(x) = x^{1+\varepsilon}$, $0 \le \varepsilon \le 1$ [1, Corollary 6].

(b) For M any holomorphically contracting metric (with $M_{\mathbb{D}}(0;1)=1$), using the fact that $\mathbb{D} \ni \zeta \mapsto (-\delta,\zeta) \in G_\psi$, $M_{G_\psi}(p_\delta;(0,1)) \le 1$. On the other hand, it follows by inclusion of domains that $M_{G_\psi} \ge M_{\mathbb{D}^2}$, therefore $M_{G_\psi}(p_\delta;(x_N,x_T)) \ge \max\{|x_N|,|x_T|\}$. In particular, $M_{G_\psi}(p_\delta;(0,1)) = 1$.

(c) If $\psi_1 \le \psi_2$, $M_{G_{\psi_1}} \ge M_{G_{\psi_2}}$. From this and Theorem 1 applied to $\psi_2(x) = x$, we deduce that if $\psi_1(x) \gtrsim x$, and $X \in \mathbb{C}^2$, then $S_{G_{\psi_1}}(p_\delta;X)$, $\kappa^{(2)}_{G_{\psi_1}}(p_\delta;X)$, and $\widetilde{\kappa}_{G_{\psi_1}}(p_\delta;X)$ are all comparable to $|X|$.

(d) The Carathéodory-Reiffen metric is the smallest of the holomorphically contracting metrics with $M_{\mathbb{D}}(0;1)=1$, so always provides a lower bound to the above quantities. Recall that, for $p \in \Omega \subset \mathbb{C}^n$ and $X \in \mathbb{C}^n$, the Carathéodory-Reiffen metric is given by

$$\gamma_\Omega(p;X) := \sup\left\{ |Df(p)\cdot X| : f \in \mathcal{O}(\Omega,\mathbb{D}), f(p)=0 \right\}.$$

If ψ is increasing, by the Hartogs phenomenon, any $f \in \mathcal{O}(G_\psi,\mathbb{D})$ extends to a holomorphic function on $D(0,\sup_{[0;1)}\psi) \times \mathbb{D}$, so $\gamma_\Omega(p_\delta;(x_N,x_T)) \le c(|x_N|+|x_T|)$. This was already pointed out in [2] for instance.

The next proposition sums up some simple facts about the Kobayashi-Royden metric taken on directions close to the "tangential" direction $(0,1)$.

Proposition 2.

(1) If $\psi(x) \geq c_0 x$ for some $c_0 > 0$ and $|x_N| \leq \min(1, c_0)|x_T|$, then

$$|x_T| \leq \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \leq \max\left(1, \frac{|x_N|}{|x_T|} \cdot \frac{1}{1-\delta}\right)|x_T|.$$

(2) If $\frac{\psi(x)}{x}$ is a strictly increasing function on $(0;1)$, if $0 < \delta \leq \delta^*$, where δ^* is the unique solution of $1 - \delta = \frac{\delta}{\psi^{-1}(\delta)}$, and if $|x_N| \leq \min\left(1, \frac{\delta}{\psi^{-1}(\delta)}\right)|x_T|$, then $\kappa_{G_\psi}(p_\delta; (x_N, x_T)) = |x_T|$.

Notice that in Part (1), when $|x_N| < |x_T|$ and δ is small enough, $\kappa_{G_\psi}(p_\delta; (x_N, x_T)) = |x_T|$. Part (2) was obtained for the case $\psi(x) = x^2$ in [1, Lemma 7(1)].

The estimates for vectors close to the normal direction $(1,0)$ are a considerably more delicate matter; [1, Proposition 3] shows that the lower estimates cannot be the same as the upper estimates. See also, for slightly different model domains, the results of Fu [4, Proposition 2.3].

Typical examples for our various situations are provided by the power functions $\psi(x) = x^\beta$, $\beta > 0$. We begin by comparing $\psi(x)$ to $x^{1/2}$.

Proposition 3.

(1) If $\psi(x) \geq c_0\sqrt{x}$ for some $c_0 > 0$, then there exists $C \geq 1$ such that

$$\max(|x_N|, |x_T|) \leq \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \leq C\max(|x_N|, |x_T|).$$

(2) If $\frac{\psi(x)}{\sqrt{x}}$ is an increasing function on $(0;1)$, there exists $c > 0$ such that

$$cF_2\left(\delta, \left|\frac{x_T}{x_N}\right|\right)|x_N| \leq \kappa_{G_\psi}(p_\delta; (x_N, x_T)),$$

where

$$F_2(\delta, t) := \min\left(\frac{\sqrt{\psi^{-1}(\delta)}}{\delta}, \frac{t}{\psi(t^2)}\right).$$

Remarks 2.

(a) The paradigm for Part (1) is given by $\psi(x) = \sqrt{x}$ and G_ψ then looks like a bidisc with a thin cuspidal wedge removed. The result suggests that such inner cusps are "too thin to be seen" by the Kobayashi-Royden metric at points within the axis of the cusp. On the other hand, when $\frac{\psi(x)}{\sqrt{x}}$ is increasing, then $\psi(x) \leq \psi(1)\sqrt{x}$.

(b) Part (2) generalizes to vectors close to the normal direction the lower bound that had been obtained in [1, Lemma 8], in the case $c_0 \leq \left|\frac{x_T}{x_N}\right| \leq \delta^{\frac{1}{\beta}-1}$, for the domain

$$\Omega_\beta := \left\{\operatorname{Re} z_1 < |z_2|^\beta + |\operatorname{Im} z_1|^\beta\right\}, \quad \beta > 1.$$

(c) Theorem 4 (1) gives a sharper result than Proposition 3 (2), under an additional hypothesis about the growth of ψ.

Theorem 4.

(1) Let $\frac{\psi(x)}{\sqrt{x}}$ be an increasing function and let $x \mapsto \psi_1(x) := \frac{\psi(x)}{x}$ be a decreasing function on $(0;1)$, with a "halving" property:
(H) there exists $K > 1$ such that for any $x \in (0; 1/K]$, $\psi_1(Kx) \leq \frac{1}{2}\psi_1(x)$.
Then there exist $0 < c < C$ such that, if $|x_T| \leq |x_N|$,

$$cF_3\left(\delta, \left|\frac{x_T}{x_N}\right|\right)|x_N| \leq \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \leq CF_3\left(\delta, \left|\frac{x_T}{x_N}\right|\right)|x_N|,$$

where

$$F_3(\delta,t) := \min\left(\frac{\sqrt{\psi^{-1}(\delta)}}{\delta}, \frac{8t}{\sqrt{\psi_1^{-1}\left(\frac{1}{8t}\right)}}\right).$$

(2) Let $\psi(x) = x$. Then there exist $0 < c < C$ such that for any $(x_N, x_T) \in \mathbb{C}^2$,

$$c\max\left(\frac{|x_N| - |x_T|}{\sqrt{\delta}}, |x_T|\right) \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le C\max\left(\frac{|x_N| - |x_T|}{\sqrt{\delta}}, |x_T|\right).$$

(3) If $\frac{\psi(x)}{x}$ is an increasing function on $(0;1)$, then there exist $C > 0$ such that if $0 < \delta < \psi(1)$, and if $|x_T| \le \frac{\psi^{-1}(\delta)}{8\delta}|x_N|$,

$$\frac{1}{4\sqrt{2}}\frac{\sqrt{\psi^{-1}(\delta)}}{\delta}|x_N| \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le C\frac{\sqrt{\psi^{-1}(\delta)}}{\delta}|x_N|.$$

Typical examples are provided by the power functions $\psi(x) = x^\beta$. For the reader's convenience, we collect the consequences of Propositions 2 and Theorem 4 for the power functions in the corollary below; the case $\beta = 1$ is already dealt with in Theorem 4 (2).

Corollary 5. *Let $\psi(x) = x^\beta$. Then there exist $0 < c < C$, depending on β, such that:*

(1) If $0 < \beta \le \frac{1}{2}$, $\max(|x_N|, |x_T|) \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le C\max(|x_N|, |x_T|)$.

(2) If $\frac{1}{2} < \beta < 1$, then:

- *if $0 \le |x_T| \le \delta^{\frac{1}{\beta}-1}|x_N|$, then $c\delta^{\frac{1}{2\beta}-1}|x_N| \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le C\delta^{\frac{1}{2\beta}-1}|x_N|$;*
- *if $\delta^{\frac{1}{\beta}-1}|x_N| \le |x_T| \le |x_N|$, then*
  $c\left|\frac{x_T}{x_N}\right|^{1-\frac{2}{1-\beta}}|x_N| \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le C\left|\frac{x_T}{x_N}\right|^{1-\frac{2}{1-\beta}}|x_N|$;
- *if $|x_N| \le |x_T| \le \frac{1}{1-\delta}|x_N|$, then $|x_T| \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le \frac{1}{1-\delta}|x_N|$;*
- *if $\frac{1}{1-\delta}|x_N| \le |x_T|$, then $\kappa_{G_\psi}(p_\delta; (x_N, x_T)) = |x_T|$.*

(3) If $1 < \beta$, then:

- *if $|x_T| \le \frac{1}{8}\delta^{\frac{1}{\beta}-1}|x_N|$, then*
  $\frac{1}{4\sqrt{2}}\delta^{\frac{1}{2\beta}-1}|x_N| \le \kappa_{G_\psi}(p_\delta; (x_N, x_T)) \le C\delta^{\frac{1}{2\beta}-1}|x_N|$;
- *if $\delta^{\frac{1}{\beta}-1}|x_N| \le |x_T|$, and if $0 < \delta \le \delta^*$, where δ^* is the unique solution of $1 - \delta = \delta^{1-\frac{1}{\beta}}$, then $\kappa_{G_\psi}(p_\delta; (x_N, x_T)) = |x_T|$.*

Remarks 3.

(a) A more precise result than ours for the case $\beta = 2$ was obtained in [1, Lemma 7, (1)], but Proposition 3 (2) and Theorem 4 (3) generalize [1, Lemma 7, (2), (3)].

(b) In Part (2), one could have considered $\psi(x) = c_0x$. However, we can reduce ourselves to $c_0 = 1$ by a linear change of coordinates and using a localization lemma for the Kobayashi-Royden metric, at the cost of changing the multiplicative constants. We omit the details.

(c) There is a gap between the ranges $8 \le \dfrac{\psi^{-1}(\delta)}{\delta}\dfrac{|x_N|}{|x_T|}$ from Part (3) and $\dfrac{\psi^{-1}(\delta)}{\delta}\dfrac{|x_N|}{|x_T|} \le 1$ from Proposition 2 (2), and it is essential, since the behavior of the Kobayashi-Royden changes radically between those two. This generalizes [1, Proposition 3 (2)]. We do not know the precise behavior of the metric in the intermediate range.

(d) The reader might be surprised that in Part (1), when $|x_T| > \frac{\psi^{-1}(\delta)}{8\delta}|x_N|$, then the bounds do not depend explicitly on δ. For instance, when $|x_N| = |x_T|$, F_3 becomes constant. This is not so surprising if we recall that the metric $\kappa_{G_\psi}(p_\delta, X)$ depends on two variables, the point p_δ and the vector X.

One can reduce this to a one-variable situation in the following way: if vectors $(x_N(\delta), x_T(\delta))$ are given, then $\kappa_{G_\psi}(p_\delta; (x_N(\delta), x_T(\delta)))$ will depend on the behavior of $|x_T(\delta)|/|x_N(\delta)|$ as

$\delta \to 0$. If it tends to 0 faster than $\frac{\psi^{-1}(\delta)}{\delta}$, then the metric will behave like $\frac{\sqrt{\psi^{-1}(\delta)}}{\delta}|x_N|$; if not, it will be some function depending on how slowly $|x_T(\delta)|/|x_N(\delta)|$ goes to 0, if it does, and bounded below if it does not.

If we apply this to the example of the function $\psi(x) = x^\beta$, with $\beta \in (\frac{1}{2}; 1)$, and a family of vectors given by $X(\delta) = (1, \delta^\gamma)$, with $\gamma > 0$, then if $\gamma \in (\frac{1}{\beta} - 1, \infty)$, applying part (1) we have

$$c\delta^{\frac{1}{2\beta}-1} \leq \kappa_{G_\psi}(p_\delta; X(\delta)) \leq C\delta^{\frac{1}{2\beta}-1}.$$

Parts (2) and (3) show that the same upper bound holds for all $\beta \geq \frac{1}{2}$.

On the other hand, if $\gamma \in (0, \frac{1}{\beta} - 1]$, we see that the rate of blow-up depends on γ (notice that the exponent is a linear interpolation between the two extreme cases):

$$c\delta^{-\gamma\frac{2\beta-1}{2(1-\beta)}} \leq \kappa_{G_\psi}(p_\delta; X(\delta)) \leq C\delta^{-\gamma\frac{2\beta-1}{2(1-\beta)}}.$$

Finally, the case of a constant vector with non-zero x_T-component (so $\gamma = 0$) would yield a bounded estimate.

The proof of Theorem 1 is given in Section 2, of Propositions 2 and 3 in Section 3, and of Theorem 4 in Section 4.

2. Proof of Theorem 1

2.1. **Upper estimate for $\kappa^{(2)}_{G_\psi}$.** Consider the map given, for $\zeta \in \mathbb{D}$, by

$$\varphi(\zeta) = \left(-\delta + \frac{\delta}{\psi^{-1}(\delta)}\zeta, \zeta\right).$$

The fact that $\mathrm{Re}(\varphi_1(\zeta)) < \psi(|\varphi_2(\zeta)|)$ is clear when $|\zeta| \leq \psi^{-1}(\delta)$; otherwise, use $\dfrac{\delta}{\psi^{-1}(\delta)} \leq \dfrac{\psi(|\zeta|)}{|\zeta|}$, which once again yields the inequality. Since $\lim_{\delta\to 0} \frac{\delta}{\psi^{-1}(\delta)} = 0$, $\varphi(\zeta) \in G_\psi$ for all $\zeta \in \mathbb{D}$ when δ is small enough.

So $\kappa_{G_\psi}(p_\delta; (1, \frac{\psi^{-1}(\delta)}{\delta})) \leq \frac{\psi^{-1}(\delta)}{\delta}$.

We saw in Remark 1(b) that $\kappa_{G_\psi}(p_\delta; (0, 1)) \leq 1$.

Since we have a Finsler metric, it is enough to obtain the desired estimate for $x_N = 1$, and

$$\kappa^{(2)}_{G_\psi}(p_\delta; (1, x_T)) \leq \kappa_{G_\psi}(p_\delta; (1, \frac{\psi^{-1}(\delta)}{\delta})) + \left|x_T - \frac{\psi^{-1}(\delta)}{\delta}\right| \kappa_{G_\psi}(p_\delta; (0, 1)) \leq 2\frac{\psi^{-1}(\delta)}{\delta} + |x_T|.$$

2.2. **Upper estimate for $\widetilde{\kappa}_{G_\psi}$.** The estimate will be established if we show that $D(0, \frac{\delta}{\psi^{-1}(\delta)}) \times \mathbb{D} \subset I_z\widetilde{\kappa}_{G_\psi}$. By the Hartogs phenomenon, this will be true if $D(0, \frac{\delta}{\psi^{-1}(\delta)}) \times \partial\mathbb{D} \subset \overline{I_z\kappa_{G_\psi}}$.

For $|a| < \frac{\delta}{\psi^{-1}(\delta)}$, $\theta \in \mathbb{R}$, consider the map given by

$$\mathbb{D} \ni \zeta \mapsto \varphi(\zeta) = \left(-\delta + a\zeta, e^{i\theta}\zeta\right).$$

Since $\mathrm{Re}(\varphi_1(\zeta)) \leq \frac{\delta}{\psi^{-1}(\delta)}|\zeta|$ and $|\varphi_2(\zeta)| = |\zeta|$, seeing that $\varphi(\zeta) \in G_\psi$ reduces to verifying (4) with $x = |\zeta|$.

2.3. **Lower estimate for S_{G_ψ}.** Since the Sibony metric verifies the triangle inequality, it will be enough to estimate it on each of the basis vectors. The estimate for $(0, 1)$ was given in Remark 1(b).

To estimate $S_{G_\psi}(p_\delta; (1, 0))$ from below, we construct a function u that will be a candidate for the supremum in the definition of the Sibony metric.

Let $C_1 := \psi^{-1}(\delta/2)$, $C_2 := C_1^2$. We define

$$f(\zeta) := C_2 \left| \frac{\zeta+\delta}{\zeta-\delta} \right|^2, \text{ then } \frac{\partial^2 f}{\partial\zeta\partial\bar\zeta}(-\delta) = \frac{C_2}{4\delta^2},$$

and then $u := e^v$, with

$$v(z_1,z_2) := \begin{cases} \max\left(\log(f(z_1)+|z_2|^2), \log|z_2|+C_3\right) - C_4, \text{ for } |z_2| \le C_1, \\ \log|z_2| + C_3 - C_4, \text{ for } |z_2| \ge C_1. \end{cases}$$

When $|z_2| \le C_1$, $z \in G_\psi$, then $\operatorname{Re} z_1 \le \psi(C_1) = \delta/2$. When $\operatorname{Re}\zeta \le \delta/2$, $\left|\frac{\zeta+\delta}{\zeta-\delta}\right|^2$ is maximal for $\operatorname{Re}\zeta = \delta/2$, and for $y \in \mathbb{R}$,

$$\left| \frac{\frac{3}{2}\delta + iy}{-\frac{1}{2}\delta + iy} \right| = \left| \frac{\delta}{\frac{1}{2}\delta - iy} + \frac{\frac{1}{2}\delta + iy}{\frac{1}{2}\delta - iy} \right| \le 2 + 1.$$

Hence for $|z_2| = C_1$, $f(z_1) + |z_2|^2 \le (9+1)C_1^2$, and if we take $C_3 > \log(10C_1)$, $\log(f(z_1)+|z_2|^2) < C_3 + \log|z_2|$, and this inequality holds in a neighborhood. This implies that v is plurisubharmonic. Finally we choose $C_4 \ge C_3 + \log|C_1|$, so that $v \le 0$ on G_ψ.

For $\operatorname{Re} z_1 < 0$, $u(z_1, 0) = f(z_1)$, so $\frac{\partial^2 u}{\partial z_1 \partial \bar z_1}(p_\delta) = \frac{\psi^{-1}(\delta/2)^2}{4\delta^2}$; passing to the square root, we see that $S_{G_\psi}(p_\delta; (1,0)) \ge \frac{\psi^{-1}(\delta/2)}{2\delta}$.

It remains to see that $\psi^{-1}(\delta) \le C\psi^{-1}(\delta/2)$. Observe that to have this, it is enough to assume that $\psi(x)/x^\gamma$ is increasing for some $\gamma > 0$. Indeed, since ψ^{-1} is an increasing function,

$$\frac{\psi^{-1}(x)}{x^{1/\gamma}} = \left(\frac{\psi^{-1}(x)^\gamma}{\psi(\psi^{-1}(x))} \right)^{1/\gamma}$$

must be decreasing. So

$$\psi^{-1}(\delta/2) = (\delta/2)^{1/\gamma} \frac{\psi^{-1}(\delta/2)}{(\delta/2)^{1/\gamma}} \ge (\delta/2)^{1/\gamma} \frac{\psi^{-1}(\delta)}{\delta^{1/\gamma}} = 2^{-1/\gamma}\psi^{-1}(\delta).$$

3. Proofs of Propositions 2 and 3

3.1. **Proof of Proposition 2.** Using the fact that $\kappa_{G_\psi}(p_\delta; (x_N, x_T)) = |x_T| \kappa_{G_\psi}(p_\delta; (\frac{x_N}{x_T}, 1))$, we may assume $x_T = 1$.

The lower estimate in Part (1) follows from Remark 1(b). For the upper estimate, it is enough to consider $\psi(x) = c_0 x$ by inclusion of domains.

When $\delta \le 1 - |x_N|$, $\max(1, |x_N|(1-\delta)^{-1}) = 1$. Consider the map

$$\varphi(\zeta) := (-\delta + x_N\zeta, \zeta).$$

The condition on x_N ensures $\varphi(\mathbb{D}) \subset \mathbb{D}^2$. For $\operatorname{Re}\zeta \le 0$, we have $\operatorname{Re}\varphi_1(\zeta) < 0$ so $\varphi(\zeta) \in G_\psi$, for $\operatorname{Re}\zeta > 0$, $|\varphi_2(\zeta)| = |\zeta| > \operatorname{Re}\varphi_1(\zeta)$.

When $\delta > 1 - |x_N|$, consider

$$\varphi(\zeta) := (-\delta + \lambda x_N \zeta, \lambda\zeta), \text{ with } \lambda = \frac{1-\delta}{|x_N|} < 1.$$

Again it is easy to check $\varphi(\mathbb{D}) \subset \mathbb{D}^2$ and for $\operatorname{Re}\zeta > 0$, $|\varphi_2(\zeta)| = \lambda|\zeta| \ge \lambda|x_N\zeta| > \operatorname{Re}\varphi_1(\zeta)$.

In Part (2), the lower estimate again holds by Remark 1(b), while the upper estimate follows from considering the map $\varphi(\zeta) := p_\delta + \zeta(x_N, 1)$: again the condition on x_N ensures $\varphi(\mathbb{D}) \subset \mathbb{D}^2$. On the other hand, $\operatorname{Re}\varphi_1(\zeta) \le -\delta + |\zeta| \frac{\delta}{\psi^{-1}(\delta)}$, so when $|\zeta| < \psi^{-1}(\delta)$, $\operatorname{Re}\varphi_1(\zeta) < 0$, and when $|\zeta| \ge \psi^{-1}(\delta)$, $\operatorname{Re}\varphi_1(\zeta) \le -\delta + \psi(|\zeta|) = -\delta + \psi(|\varphi_2(\zeta)|)$.

3.2. **Proof of Proposition 3.** The lower estimate in Part (1) follows from Remark 1(b). For the upper estimate, by inclusion, it is enough to prove it for $\psi(x) = c_0 x^{1/2}$. Proposition 2 (1) takes care of the case $|x_T| > \max(1, c_0^{-1})|x_N|$.

Using the fact that $\kappa_{G_\psi}(p_\delta; (x_N, x_T)) = |x_N|\kappa_{G_\psi}(p_\delta; (1, \frac{x_T}{x_N}))$, we may assume $x_N = 1$.

Define
$$\varphi(\zeta) := (-\delta + \zeta, x_T\zeta + \frac{x_T}{c_0^2|x_T|}\zeta^2),$$
with the convention that $\frac{x_T}{|x_T|} = 1$ when $x_T = 0$.

Since we may assume $|x_T| \le \frac{1}{c_0}$, we see that $\varphi(\zeta) \in \mathbb{D}^2$ when $|\zeta| \le \min(1-\delta_0, c_0\frac{-1+\sqrt{5}}{2})$. When $\operatorname{Re}\zeta < 0$, $\operatorname{Re}\varphi_1(\zeta) < \psi(|\varphi_2(\zeta)|)$ trivially. When $\operatorname{Re}\zeta \ge 0$,
$$\psi(|\varphi_2(\zeta)|) = \psi\left(\left|\frac{x_T}{|x_T|}\zeta\right|\left||x_T| + \frac{1}{c_0^2}\zeta\right|\right) \ge \psi(\frac{1}{c_0^2}|\zeta|^2) = |\zeta| > \operatorname{Re}\varphi_1(\zeta).$$
So by an easy change of variable, $\kappa_{G_\psi}(p_\delta; (1, x_T)) \le \max((1-\delta_0)^{-1}, \frac{1+\sqrt{5}}{2c_0}) =: C_1$.

To prove Part (2), since the case $x_N = 0$ is covered by Remark 1(b), again we may assume $x_N = 1$.

Lemma 6. *Let $\varphi = (\varphi_1, \varphi_2)$ be a holomorphic map from $\mathbb{D}$ to G_ψ with $\varphi(0) = p_\delta$ and $\varphi'(0) = \lambda(1, x_T)$, with $\lambda > 0$.*

Let $0 < r \le 1$, and $M := \psi(r(\lambda|x_T| + r))$. Then
$$\lambda \le \frac{2(M+\delta)}{r}. \tag{3}$$

Proof. Let $\varphi_2(\zeta) = \zeta h(\zeta)$, with $h(\mathbb{D}) \subset \mathbb{D}$ and $h(0) = \lambda x_T$. This implies
$$|h(\zeta)| \le \frac{|h(0)| + |\zeta|}{1 + |h(0)||\zeta|} \le \lambda|x_T| + |\zeta|.$$

It follows that $\max_{|\zeta|\le r}|\varphi_2(\zeta)| \le \psi(r(\lambda|x_T|+r))$, and so $\operatorname{Re}\varphi_1(\zeta) < M$ for $|\zeta| < r$. Then elementary computations show that the function $f(\zeta) := \dfrac{\varphi_1(\zeta)+\delta}{\varphi_1(\zeta) - 2M - \delta}$ verifies $f(0) = 0$, $f(D(0,r)) \subset \mathbb{D}$, and $|f'(0)| = \left|\dfrac{\varphi_1'(0)}{2M+2\delta}\right|$. From the Schwarz Lemma, we deduce (3). □

Observe that if $\frac{\psi(x)}{\sqrt{x}}$ is an increasing function on $(0;1)$, then $\frac{\sqrt{\psi^{-1}(\delta)}}{\delta} \le \frac{t}{\psi(t^2)}$ if and only if $t \le \sqrt{\psi^{-1}(\delta)}$. So we may split the proof of Proposition 3 (2) into two cases.

Case $|x_T| \le \sqrt{\psi^{-1}(\delta)}$.

Let $r := \frac{1}{2}\sqrt{\psi^{-1}(\delta)}$. Since $\varphi_1(\mathbb{D}) \subset \mathbb{D}$, $\lambda \le 1$ so $|\varphi_2(\zeta)| \le \psi^{-1}(\delta)$, so $M \le \delta$ and Lemma 6 implies $\lambda \le 8\delta/\sqrt{\psi^{-1}(\delta)}$, q.e.d.

Case $|x_T| \ge \sqrt{\psi^{-1}(\delta)}$.

Let $r := \frac{1}{2}|x_T|$, so $M \le \psi(|x_T|^2)$, so it follows from Lemma 6 that $\lambda \le 4(\delta + \psi(|x_T|^2))/|x_T| \le 8\psi(|x_T|^2)/|x_T|$ by the hypothesis on $|x_T|$.

4. Proof of Theorem 4

4.1. **Lower estimates.** We use the same method as in the proof of Proposition 3 (2). We assume $x_N = 1$ and let φ be as in Lemma 6, r to be chosen, $\lambda > 0$; we need to bound λ from above.

Part (1). We begin by analysing the function F_3.

Lemma 7. *If $\frac{\psi(x)}{\sqrt{x}}$ is an increasing function on $(0;1)$, if furthermore ψ_1 is a decreasing function on $(0;1)$, then $\frac{\sqrt{\psi^{-1}(\delta)}}{\delta} \le \frac{8t}{\sqrt{\psi_1^{-1}\left(\frac{1}{8t}\right)}}$ if and only if $t \le \frac{\psi^{-1}(\delta)}{8\delta}$.*

Proof. First notice that $\delta \mapsto \dfrac{\sqrt{\psi^{-1}(\delta)}}{\delta} = \dfrac{\sqrt{\psi^{-1}(\delta)}}{\psi\left(\psi^{-1}(\delta)\right)}$ is a decreasing function, so the desired inequality will hold when $\delta \geq \Delta(t)$, where $\Delta(t)$ stands for the value achieving equality between the two quantities.

Now if $t = \frac{\psi^{-1}(\delta)}{8\delta} = \frac{1}{8\psi_1(\psi^{-1}(\delta))}$, then

$$\psi_1^{-1}\left(\frac{1}{8t}\right) = \psi^{-1}(\delta), \text{ so } \frac{8t}{\sqrt{\psi_1^{-1}\left(\frac{1}{8t}\right)}} = \frac{\psi^{-1}(\delta)}{\delta\sqrt{\psi^{-1}(\delta)}} = \frac{\sqrt{\psi^{-1}(\delta)}}{\delta}.$$

So $\Delta^{-1}(\delta) = \frac{\psi^{-1}(\delta)}{8\delta}$, and since $\delta \mapsto \dfrac{\psi^{-1}(\delta)}{\delta} = \dfrac{\psi^{-1}(\delta)}{\psi\left(\psi^{-1}(\delta)\right)}$ is an increasing function under the hypothesis of Part (1) of Theorem 4, $t \mapsto \Delta(t)$ is increasing as well, and $\delta \geq \Delta(t)$ is equivalent to $\frac{\psi^{-1}(\delta)}{8\delta} \geq t$. $\square$

Lemma 7 allows us to split the proof into two cases.

Case $|x_T| \leq \frac{\psi^{-1}(\delta)}{8\delta}$.

Let $r := \sqrt{\frac{\psi^{-1}(\delta)}{2}}$.

If $\lambda|x_T| \leq r$, then for $|\zeta| < r$, $|\varphi_2(\zeta)| < 2r^2$, so $\psi(|\varphi_2(\zeta)|) < \delta$. From Lemma 6 we get $\lambda \leq 4\delta/r = 4\sqrt{2}\delta/\sqrt{\psi^{-1}(\delta)}$.

If $\lambda|x_T| \geq r$, then for $|\zeta| < r$, $|\varphi_2(\zeta)| < 2r\lambda|x_T|$, so $M \leq \psi(2r\lambda|x_T|)$, so by Lemma 6

$$\lambda \leq 2\frac{\psi(2r\lambda|x_T|)}{r} \leq 2\frac{\psi(2\frac{r}{8\delta}\lambda\psi^{-1}(\delta))}{r} \leq 2\frac{r}{4\delta}\lambda\frac{\psi(\psi^{-1}(\delta))}{r} = \frac{\lambda}{2},$$

if we assume $\frac{r}{4\delta}\lambda \geq 1$, since $\frac{\psi(x)}{x}$ is decreasing; but this is a contradiction, so $\lambda \leq \frac{4\delta}{r}$ once again.

Case $\frac{\psi^{-1}(\delta)}{8\delta} \leq |x_T| \leq 1$.

Let $r := \sqrt{\psi_1^{-1}\left(\frac{1}{|x_T|}\right)}$. If $\lambda|x_T| \leq r$, then we are done. So assume $\lambda|x_T| \geq r$, then for $|\zeta| \leq r$, then $M \leq \psi(2r\lambda|x_T|)$, and by Lemma 6, $\lambda r \leq 2(\delta + \psi(2r\lambda|x_T|))$.

Since $|x_T| \geq \frac{\psi^{-1}(\delta)}{8\delta} = \frac{1}{8\psi_1(\psi^{-1}(\delta))}$, by hypothesis (H) we have

$$\begin{aligned}\delta &\leq \psi\left(\psi_1^{-1}\left(\frac{1}{8|x_T|}\right)\right) \leq \psi\left(K^3\psi_1^{-1}\left(\frac{1}{|x_T|}\right)\right)\\ &\leq \max(\frac{K^3}{2},1)\psi\left(2\psi_1^{-1}\left(\frac{1}{|x_T|}\right)\right) \leq \max(\frac{K^3}{2},1)\psi(2r^2) \leq \max(\frac{K^3}{2},1)\psi(2r\lambda|x_T|).\end{aligned}$$

So we have $\lambda r \leq C_3\psi(2r\lambda|x_T|)$, so $1 \leq 2C_3|x_T|\psi_1(2r\lambda|x_T|)$, equivalently, choosing $m \in \mathbb{N}$ such that $2^m \geq 2C_3$,

$$\psi_1(2r\lambda|x_T|) \geq \frac{1}{2C_3}\psi_1\left(\psi_1^{-1}\left(\frac{1}{|x_T|}\right)\right) \geq \frac{2^m}{2C_3}\psi_1\left(K^m\psi_1^{-1}\left(\frac{1}{|x_T|}\right)\right) \geq \psi_1\left(K^m\psi_1^{-1}\left(\frac{1}{|x_T|}\right)\right),$$

so $2r\lambda|x_T| \leq K^m r^2$, thus $\lambda \leq \frac{K^m}{2|x_T|}r$, q.e.d.

Part (3). If $|\lambda| \leq 4\delta$, the conclusion is already obtained since $\psi^{-1}(\delta) \leq 1$. Hence we may choose $r = 4\delta/|\lambda|$ in Lemma 6, and get that if there is a map φ as in that Lemma,

$$2\delta = \frac{r|\lambda|}{2} \leq \delta + \psi(r^2 + |\lambda x_T|r),$$

thus $\delta \leq \psi(r^2 + |\lambda x_T|r)$, so, since ψ is increasing,

$$\psi^{-1}(\delta) \leq r^2 + |\lambda x_T|r \leq 2\max(r^2, 4\delta|x_T|).$$

However the hypothesis implies that $8\delta|x_T| < \psi^{-1}(\delta)$, so we must have $\psi^{-1}(\delta) \le 2r^2$, which means $\frac{1}{|\lambda|} \ge \frac{1}{4\sqrt{2}}\frac{\sqrt{\psi^{-1}(\delta)}}{\delta}$.

Part (2).

Proposition 2 already covers the case where $|x_T| \ge |x_N|$. So we will assume $|x_T| \le |x_N|$, and by homogeneity, $x_N = 1$.

Take φ as in Lemma 6. We must have $\lambda \le 1$ by the Schwarz Lemma applied to φ_1. For $\zeta \in \mathbb{D}$, $\varphi_1(\zeta) = -\delta + \lambda\zeta + \zeta^2\tilde{\varphi}_1(\zeta)$, $\tilde{\varphi}_1 \in \mathcal{O}(\mathbb{D})$ and

$$|\tilde{\varphi}(\zeta)| \le \sup_{|\zeta|=1} |\varphi_1(\zeta) + \delta - \lambda\zeta| \le 1 + \delta + \lambda \le 3.$$

Thus $\operatorname{Re}\varphi_1(\zeta) \ge -\delta + \lambda \operatorname{Re}\zeta - 3|\zeta|^2$.

On the other hand, by the proof of Lemma 6, $|\varphi_2(\zeta)| \le \lambda|x_T||\zeta| + |\zeta|^2$. Applying the condition $\operatorname{Re}\varphi_1(\zeta) \le |\varphi_2(\zeta)|$ to the value $\zeta = \sqrt{\delta}$, we obtain

$$\lambda\sqrt{\delta} - 4\delta < \lambda|x_T|\sqrt{\delta} + \delta, \text{ i.e. } \lambda < \frac{5\sqrt{\delta}}{1 - |x_T|},$$

so we obtain the conclusion with $c = \frac{1}{5}$.

4.2. **Upper estimates.** For those estimates, we need to construct maps $\varphi : \mathbb{D} \longrightarrow G_\psi$ with $\varphi'(0) = \lambda(1, x_T)$ and λ as large as we can (which is still a quantity tending to 0 as δ tends to 0).

Part (1). By Lemma 7, we may split the proof into two cases.

Case $|x_T| \le \frac{\psi^{-1}(\delta)}{\delta}$.

Let $\lambda := \frac{\delta}{2\sqrt{\psi^{-1}(\delta)}}$, and consider the map given, for $\zeta \in \mathbb{D}$, by

$$\varphi(\zeta) = \left(-\delta + \lambda\zeta, \lambda x_T\zeta + \frac{\zeta^2}{2}\right).$$

It is enough to show that $\varphi(\mathbb{D}) \subset G_\psi$.

First note that $\lambda|x_T| \le \frac{1}{2}\sqrt{\psi^{-1}(\delta)}$, so

$$|\varphi_2(\zeta)| \le \frac{1}{2}\sqrt{\psi^{-1}(\delta)}|\zeta| + \frac{|\zeta|^2}{2}.$$

When $|\zeta| < 2\sqrt{\psi^{-1}(\delta)}$, then $\operatorname{Re}(\lambda\zeta) \le \lambda|\zeta| < \delta$, so $\operatorname{Re}\varphi_1(\zeta) < 0$ and $\varphi(\zeta) \in G_\psi$.

Suppose now $|\zeta| \ge 2\sqrt{\psi^{-1}(\delta)}$. Then

$$\left|\lambda x_T\zeta + \frac{\zeta^2}{2}\right| \ge \frac{|\zeta|^2}{2} - \frac{1}{4}|\zeta|^2 = \frac{|\zeta|^2}{4}.$$

So in this case it will be enough to prove

$$\frac{\delta}{2\sqrt{\psi^{-1}(\delta)}}|\zeta| < \psi\left(\frac{|\zeta|^2}{4}\right) + \delta,$$

for which it is enough to have

$$\frac{\psi\left((\frac{|\zeta|}{2})^2\right)}{\frac{|\zeta|}{2}} \ge \frac{\delta}{\sqrt{\psi^{-1}(\delta)}} = \frac{\psi((\sqrt{\psi^{-1}(\delta)})^2)}{\sqrt{\psi^{-1}(\delta)}},$$

which is satisfied since $|\zeta|/2 \ge \sqrt{\psi^{-1}(\delta)}$, and $\frac{\psi(x^2)}{x} = \frac{\psi(x^2)}{\sqrt{x^2}}$ is increasing.

Notice that in this case, we have not used the hypothesis that ψ_1 be decreasing, so that this also proves the required upper estimate for Part (3).

Case $|x_T| \ge \frac{\psi^{-1}(\delta)}{8\delta}$.

Assume $\delta \leq \frac{1}{2}$. It will be enough to prove that $\varphi(\zeta) \in G_\psi$ for $|\zeta| < 1-\delta$ and

$$\varphi(\zeta) := (-\delta + \lambda\zeta, \lambda x_T \zeta + \frac{x_T}{|x_T|}\zeta^2), \text{ with } \lambda := \frac{1}{|x_T|}\sqrt{\psi_1^{-1}\left(\frac{1}{|x_T|}\right)}.$$

Note that $\lambda \leq \psi(1)$ so that $\varphi(\zeta) \in \mathbb{D}^2$ for $|\zeta| \leq \frac{1}{2}\min(1, \frac{1}{\psi(1)})$.

Indeed, for $y \geq 1$,

$$\psi(1) = \frac{\psi(1^2)}{1} \geq \frac{\psi(y^{-2})}{y^{-1}} = \frac{\psi_1(y^{-2})}{y},$$

so, since ψ_1 is decreasing, $\psi_1^{-1}(\psi(1)y) \leq y^{-2}$. Now apply this to $y = \frac{1}{\psi(1)|x_T|}$.

For $\operatorname{Re}\zeta \leq 0$, clearly $\varphi(\zeta) \in G_\psi$. For $\operatorname{Re}\zeta > 0$,

$$|\varphi_2(\zeta)| \geq \left|\frac{x_T}{|x_T|}\zeta\right| |\lambda|x_T| + \zeta| \geq |\zeta| \max(|\zeta|, \lambda|x_T|).$$

On the other hand, $\operatorname{Re}\varphi_1(\zeta) \leq \lambda|\zeta|$.

Subcase 1. $|\zeta| \leq \lambda|x_T|$.

It is enough to verify $\lambda|\zeta| \leq \psi(\lambda|x_T||\zeta|)$, i.e. $\frac{1}{|x_T|} \leq \psi_1(\lambda|x_T||\zeta|)$. But ψ_1 is a decreasing function, so it is enough to check this for $|\zeta| = \lambda|x_T|$, where we obtain

$$\psi_1(\lambda^2|x_T|^2) = \psi_1(\psi_1^{-1}\left(\frac{1}{|x_T|}\right)) = \frac{1}{|x_T|}.$$

Subcase 2. $|\zeta| \geq \lambda|x_T|$.

It is enough to verify $\lambda|\zeta| \leq \psi(|\zeta|^2)$, but since $\frac{\psi(x)}{\sqrt{x}}$ is an increasing function,

$$\frac{\psi(|\zeta|^2)}{|\zeta|} \geq \frac{\psi(\lambda^2|x_T|^2)}{\lambda|x_T|} = \lambda|x_T|\psi_1(\lambda^2|x_T|^2) = \lambda.$$

Part (2).

This proof will gather piecemeal results on a number of different cases. We would like to find a more elegant argument.

Proposition 2 already covers the case where $|x_T| \geq |x_N|$. On the other hand, Part (3) proves in particular the lower estimate for the case $|x_T| \leq \frac{1}{8}|x_N|$. So we will assume $\frac{1}{8}|x_N| < |x_T| < |x_N|$, and by homogeneity, $x_N = 1$.

Case 1. $0 < \delta < (1-|x_T|)^2$.

Define

$$\alpha := \frac{(1-|x_T|)^2}{2\delta}, \quad \varphi(\zeta) := \left(-\delta + \zeta, x_T\zeta + \alpha\frac{x_T}{|x_T|}\zeta^2\right).$$

We claim that $\varphi(\zeta) \in G_\psi$ for $|\zeta| < \frac{1}{2}\sqrt{\delta}(1-|x_T|)^{-1}$, which will prove the estimate.

First note that $|\varphi_1(\zeta)| \leq \delta + \frac{1}{2} < 1$ since $\delta < \frac{1}{2}$, and $|\varphi_2(\zeta)| \leq \frac{|x_T|\sqrt{\delta}}{2(1-|x_T|)} + \frac{1}{8} \leq \frac{1}{2} + \frac{1}{8} < 1$.

We must still check the condition $\operatorname{Re}\varphi_1(\zeta) < |\varphi_2(\zeta)|$. For $\operatorname{Re}\zeta < \delta$, this is immediate. For $x := \operatorname{Re}\zeta \geq \delta > 0$, note that

$$|\varphi_2(\zeta)| = |\zeta|\,||x_T| + \alpha\zeta| \geq x(|x_T| + \alpha x),$$

so that the required inequality will be satisfied if

$$x(|x_T|+\alpha x)-(x-\delta) = \alpha x^2-(1-|x_T|)x+\delta = \frac{1}{2\delta}\left(((1-|x_T|)x)^2 - 2\delta(1-|x_T|)x + 2\delta^2\right) > 0,$$

which clearly holds.

Case 2. $\delta \geq (1-|x_T|)^2$.

First notice that since $\frac{1}{8} \leq |x_T| \leq 1$ and $\frac{1-|x_T|}{\sqrt{\delta}} \leq 1$, $\frac{1}{8} \leq \max\left(\frac{|x_N|-|x_T|}{\sqrt{\delta}}, |x_T|\right) \leq 1$, so it will be enough to find a map $\varphi \in \mathcal{O}(\mathbb{D}, G_\psi)$ with $\varphi(0) = p_\delta$ and $\varphi'(0) = \lambda(1, x_T)$, with λ some nonzero constant. Let

$$\varphi(\zeta) := \left(-\delta + \lambda\zeta, \lambda x_T \zeta + \frac{1}{4}\frac{x_T}{|x_T|}\zeta^2\right).$$

We claim that $\varphi(\mathbb{D}) \subset G_\psi$ for $\lambda = \min(1-\delta_0, \frac{1}{8})$. Such a choice of λ in particular implies that $\varphi(\mathbb{D}) \subset \mathbb{D}^2$.

Let $\zeta = x + iy$, $x, y \in \mathbb{R}$. Note first that $\operatorname{Re}\varphi_1(\zeta) < 0$ when $x < \delta/\lambda$, so that it is enough to look at values of ζ with $\lambda x \geq \delta$. We then need to check $(\operatorname{Re}\varphi_1(\zeta))^2 < |\varphi_2(\zeta)|^2$, i.e.

$$(\lambda x - \delta)^2 < (x^2+y^2)\left((\lambda|x_T| + \frac{x}{4})^2 + \frac{y^2}{16}\right).$$

This will hold for all $\zeta \in \mathbb{D}$ if and only if it holds for all $x \in (-1;1)$, $y = 0$, so we need to check

$$\lambda^2 x^2 (1-|x_T|^2) + \delta^2 < 2\lambda x\delta + \lambda|x_T|\frac{x^3}{2} + \frac{x^4}{16}.$$

The left hand side is dominated by $2\lambda^2 x^2\sqrt{\delta} + \lambda x\delta$, so it is enough to check

$$2\lambda^2 x^2\sqrt{\delta} < \lambda x\delta + \lambda|x_T|\frac{x^3}{2} + \frac{x^4}{16}.$$

Case 2.1. $x < 4\sqrt{\delta}$.

It will be enough to verify $8\lambda^2 x\delta \leq \lambda x\delta$, which follows from $\lambda \leq \frac{1}{8}$.

Case 2.2. $x \geq 4\sqrt{\delta}$.

Then

$$\lambda|x_T|\frac{x^3}{2} \geq \lambda\frac{x^3}{16} \geq \frac{1}{4}\lambda x^2\sqrt{\delta} \geq 2\lambda^2 x^2\sqrt{\delta},$$

whenever $\lambda \leq \frac{1}{8}$. Since $\frac{x^4}{16} > 0$, this finishes the proof.

Part (3). The case $|x_T| \leq \frac{\psi^{-1}(\delta)}{\delta}$ has been covered by the proof of Part (1), so we may assume $|x_T| \geq \frac{\psi^{-1}(\delta)}{\delta}$.

Consider the map given, for $\zeta \in \mathbb{D}$, by

$$\varphi(\zeta) = \left(-\delta + \frac{1}{x_T}\zeta, \zeta\right).$$

We want to see that $\varphi(\zeta) \in G_\psi$ when $|\zeta| < c\frac{\delta}{\sqrt{\psi^{-1}(\delta)}}$. This condition implies that $|\zeta|/|x_T| \leq \frac{\delta|\zeta|}{\psi^{-1}(\delta)} < 1$. It is thus enough to see that

$$-\delta + \operatorname{Re}\left(\frac{1}{x_T}\zeta\right) \leq -\delta + \frac{\delta}{\psi^{-1}(\delta)}|\zeta| < \psi(|\zeta|). \tag{4}$$

This is obviously satisfied for $|\zeta| \leq \psi^{-1}(\delta)$; for $|\zeta| > \psi^{-1}(\delta)$, since $\frac{\psi(x)}{x}$ is increasing,

$$\frac{\delta}{\psi^{-1}(\delta)} \leq \frac{\psi(|\zeta|)}{|\zeta|} < \frac{\psi(|\zeta|)}{|\zeta|} + \frac{\delta}{|\zeta|},$$

which proves the property.

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N. Nikolov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev 8, 1113 Sofia, Bulgaria

Faculty of Information Sciences, State University of Library Studies and Information Technologies, Shipchenski prohod 69A, 1574 Sofia, Bulgaria

Email address: nik@math.bas.bg

Thomas, Pascal J., Univ Toulouse, INSA Toulouse, CNRS, IMT, Toulouse, France

Email address: pascal.thomas@math.univ-toulouse.fr