MOVING RECTANGULAR SOFAS IN PLANAR AND SPATIAL CORRIDORS

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Abstract. We consider eight natural planar corridors, including the standard L-shaped one, and characterize the rectangles that can move around their corners. As a bi-product we describe completely the corresponding rectangles with maximum area, as well as the rectangular parallelepipeds with maximum volume that can move around the corners of the spatial analogues of the considered eight planar corridors.

1. Introduction

A well-known version of the famous *moving sofa problem* [4, 2, 7, 1] is to characterize the rectangles that can move around the right-angled corner in an L-shaped corridor with given widths. This problem has been particularly solved in [6] (cf. also [3] and [5]), where the longest such rectangle of a given width has been found in terms of the smallest positive root of a six degree algebraic equation.

The goal of this paper is to consider eight planar corridors, including the standard L-shaped one, and to completely solve the aforementioned problem for each of them. More precisely, let

$$\mathcal{D} = \{x = a, 0 < y < b\}\ ,\ \mathcal{A}_0 = \mathcal{D} \cup \{0 < x < a\}\ ,\ \mathcal{B}_0 = \mathcal{D} \cup \{x > a\},$$

$$\mathcal{A}_1 = A_0 \cap \{y > 0\}\ ,\ \mathcal{B}_1 = B_0 \cap \{y > 0\}\ ,\ \mathcal{B}_2 = \mathcal{B}_0 \cap \{y < b\},\ \mathcal{B}_3 = \mathcal{B}_0 \cap \{0 < y < b\}.$$

Given a pair $(i, j) \in \{0, 1\} \times \{0, 1, 2, 3\}$ we denote by $\mathcal{R}_{ij}$ the set of open rectangles that can move around the corner in the corridor $\mathcal{C}_{ij} = \mathcal{A}_i \cup \mathcal{B}_j$.[1] (Fig. 1)

For arbitrary $a, b, c > 0$ let $m(a, b, c) = \min_\Delta f_{(a,b,c)}$, where $\Delta = [0, \pi/2]$ and

$$f_{(a,b,c)}(t) = a \sin t + b \cos t - c \sin t \cos t. \tag{1}$$

Set also $l = \sqrt{a^2 + b^2}$, $h = ab/\sqrt{a^2 + b^2}$ and $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.

Our main result is the following theorem which characterizes completely the rectangles from the sets $\mathcal{R}_{ij}$ and shows that these sets are divided into four nonintersecting groups.

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[1]For a rectangle R in the plane this means that there is a continuous path Φ_t in the Euclidean group $E(2)$ of rigid motions such that $\Phi_0(R) \subset \mathcal{A}$, $\Phi_1(R) \subset \mathcal{B}$ and $\Phi_t(R) \subset \mathcal{C}_{ij}$ for any $t \in [0, 1]$.

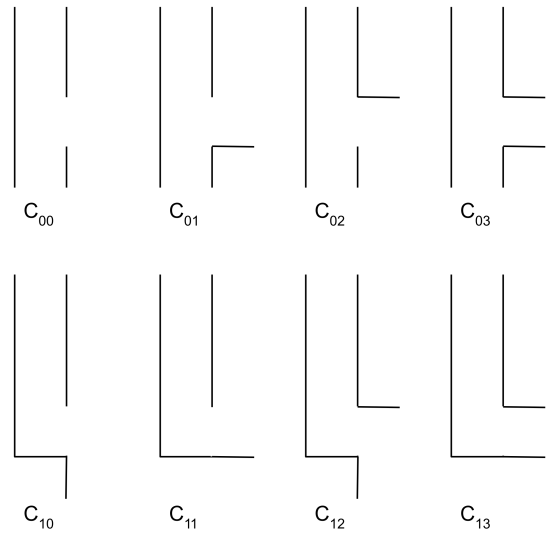


FIGURE 1. The corridors $\mathcal{C}_{ij}$.

Theorem 1. *The following identities hold true:* $\mathcal{R}_{00} = \mathcal{R}_{01} = \mathcal{R}_{02}$*,* $\mathcal{R}_{10} = \mathcal{R}_{12}$*,* $\mathcal{R}_{11} = \mathcal{R}_{13}$*,* $\mathcal{R}_{00} = \mathcal{R}_{10} \cup \mathcal{R}_{03}$*,* $\mathcal{R}_{13} = \mathcal{R}_{10} \cap \mathcal{R}_{03}$*. Moreover, a rectangle with side lengths* c *and* $d, c \geq d$*, belongs to the set:*
(i) $\mathcal{R}_{0i}, i = 0, 1, 2,$ *if and only if* $d \leq a \wedge b, cd \leq a$*;*
(ii) $\mathcal{R}_{03}$*, if and only if*
(ii$_1$) $h \leq d \leq a \wedge b, cd \leq ab$*, or (ii$_2$)* $d \leq h \wedge m(a, b, c)$*;*
(iii) $\mathcal{R}_{1i}, i = 0, 2,$ *if and only if*
(iii$_1$) $d \leq h, cd \leq ab$*, (iii$_2$)* $h \leq d \leq a \wedge b, c \leq a \vee b$*, or (iii$_3$)* $h \leq d \leq m(a, b, c)$*;*
(iv) $\mathcal{R}_{1i}, i = 1, 3$ *if and only if*
(iv$_1$) $c \leq a \vee b, d \leq a \wedge b$*, or (iv$_2$)* $d \leq m(a, b, c)$*.*

Note that the above conditions do not depend on the position of the rectangle R in the corridor A_i. Observe also that the case $d = 0$ in Theorem 1 corresponds to the so-called *ladder problem* for L-shaped corridors [3]. Its solution for the corridors $\mathcal{C}_{ij}$ is given by

Corollary 2. *The length of the longest line segment that can move around the corner in the corridor* $\mathcal{C}_{03}, \mathcal{C}_{11}$*, or* $\mathcal{C}_{13}$ *is* $(a^{2/3} + b^{2/3})^{3/2}$*, whereas for the remaining five corridors, this can be done for a segment of arbitrary length.*

Another consequence of Theorem 1 (using inequality (4)) is the following

Corollary 3. *The rectangles from the sets* $\mathcal{R}_{ij}$ *have maximum area* ab*. Moreover, the maximum area rectangles from the set:*
(i) $\mathcal{R}_{0i}, i = 0, 1, 2,$ *have side lengths* ab/d *and* d*, where* $0 < d \leq a \wedge b$*;*
(ii) $\mathcal{R}_{03}$ *have side lengths* ab/d *and* d*, where* $h \leq d \leq a \wedge b$*;*
(iii) $\mathcal{R}_{1i}, i = 0, 2,$ *have side lengths* a *and* b*, or* ab/d *and* d*, where* $0 < d \leq h$*;*
(iv) $\mathcal{R}_{1i}, i = 1, 3,$ *have side lengths* a *and* b*, or* l *and* h*.*

In Section 4 of the paper we consider the spatial corridors $\mathcal{S}_{ij} = \mathcal{C}_{ij} \times (0, c), c > 0$, and describe completely the rectangular parallelepipeds with maximum volume that can move around the corner in $\mathcal{S}_{ij}$. Denoting the set of these parallelepipeds by $\mathcal{P}_{ij}$ we prove the following theorem which gives a partial answer to [6, Question 4, p. 200].

Theorem 4. *Any $P \in \mathcal{P}_{ij}$ has volume abc and $P = R \times (0, c)$ for a rectangle $R \in \mathcal{R}_{ij}$ with maximum volume.*

As a consequence of Theorem 4 and Corollary 2 we obtain the following solution of the *spatial leader problem.*

Corollary 5. *The length of the longest line segment that can move around the corner in the corridor $\mathcal{S}_{03}, \mathcal{S}_{11}$, or $\mathcal{S}_{13}$ is $((a^{2/3} + b^{2/3})^3 + c^2)^{1/2}$, whereas for the remaining five corridors, this can be done for a segment of arbitrary length.*

The paper is organized as follows. In Section 2 we characterize the rectangles from the sets $\mathcal{R}_{00}$ and $\mathcal{R}_{13}$ (Proposition 7 and Proposition 8, respectively) and in Section 3 we use these results to prove Theorem 1. Let us mention two difficulties that arise here. The first is the justification of the corresponding necessary conditions, since when moving a rectangle in a corridor it can rotate around the corner in either direction. Note that in [6] and [3] the case of "anti-rotation" (see the proof of Proposition 8) has not been considered and a rigorous proof of the necessary condition for L-shaped corridors is missing. The second difficulty is that for proving the sufficiency of the conditions for the side lengths of a rectangle in R_{03} one has to combine the movements used for rectangles in $\mathcal{R}_{00}$ and $\mathcal{R}_{13}$.

The proof of Theorem 4, given in Section 4, is two-step. We first prove the inequality $V_P \leq abc$, for a rectangular parallelepiped P that can move around the corner of a corridor S_{ij}. This is done by analyzing the position of P at the last moment until at least six of its vertices lie in $\mathcal{A}_i \times (0, c)$. Then we prove that if $V_P = abc$, then $P = R \times (0, c)$ for a rectangle $R \in \mathcal{R}_{ij}$ with maximum volume. This follows by combining the above analysis and the following

Lemma 6. *Let α and β be parallel planes in the space and P be a rectangular parallelepiped with two opposite faces on these planes. Then at any moment of a movement of P in the layer between α and β these faces lie on them.*

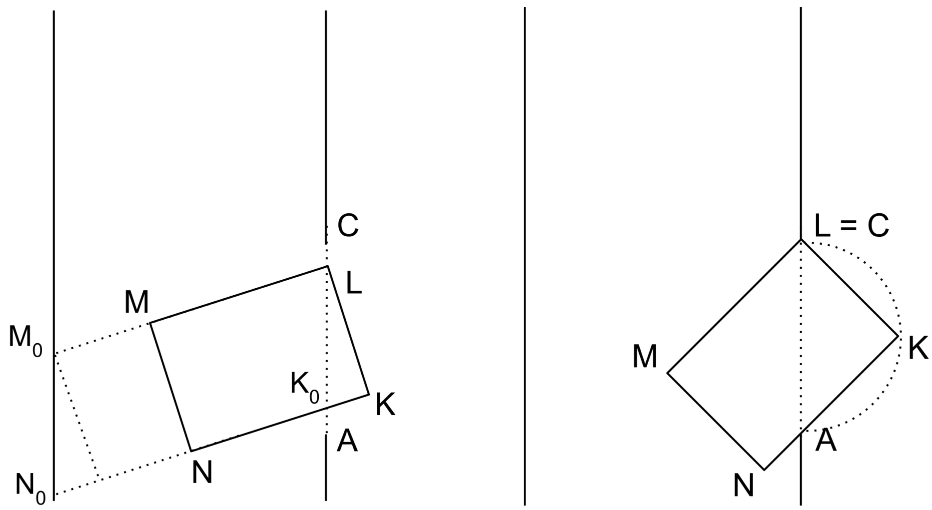
The Appendix at the end of the paper collects some technical facts used in the proofs of the results stated in Section 1.

2. Moving rectangles in the corridors $\mathcal{C}_{00}$ and $\mathcal{C}_{13}$

To prove Theorem 1 we first characterize the rectangles from the sets $\mathcal{R}_{00}$ and $\mathcal{R}_{13}$.

Proposition 7. *A rectangle with side lengths c and d, $c \geq d$, belongs to the set $\mathcal{R}_{00}$ if and only if $d \leq a \wedge b$ and $cd \leq ab$.*

Proof. We first prove the *only if part* of the proposition. Let $A = (a, 0)$ and $C = (a, b)$ be the ends of the segment $\mathcal{D}$ and $R = KLMN$ be a rectangle with $|KL| = d$ and



(A) (B)

FIGURE 2. The corridor $\mathcal{C}_{00}$.

$|KN| = c$. Note that if $d > a$, then $R \not\subseteq \mathcal{A}_0$ and if $d > b$, then it is not possible to move R into $\mathcal{B}_0$ trough $\mathcal{D}$. Hence $d \le a \wedge b$.

Suppose now that it is possible to move R from $\mathcal{A}_0$ to $\mathcal{B}_0$ trough $\mathcal{D}$. Consider the first moment when at least two vertices of R lie in $\mathcal{B}_0$. Suppose first that at this moment there is no vertex of R lying in $\{x > a\}$. Then its side is lying on the segment $\mathcal{D}$ and it is clear that $cd \le ab$. The other possibility is that there is only one vertex of R in $\{x > a\}$, say K. Then either L or M, say L, lies on $\mathcal{D}$. We can assume that the line KN meets $\mathcal{D}$ and the line $\{x = 0\}$ at some points K_0 and N_0, respectively, and let the line LM meets $\{x = 0\}$ at M_0 (Fig. 2a).

Then

$$ab \ge a|K_0L| = S_{K_0LM_0N_0} \ge S_{KLMN} = cd$$

and the *only if* part is proved.

Conversely, let $c \ge d, d \le a \wedge b$ and $cd \le ab$. Then it is easy to see that for any position of R in $\mathcal{A}_0$ one can move it in a way that one of its longest sides, say KN, lies on the line $\{x = a\}, K = C$ and N is below the point C. If $c \le b$, then one can move R to $\mathcal{B}_0$ by a translation parallel to the line $\{y = 0\}$. Let now $c > b$. Then one moves R such that the vertex K is moving on the semicircle with diameter AC, lying in the half-plane $\{x > a\}$. Consider the moment when $|KC| = d$, i.e. $L = C$ (Fig. 2b). We will show that then $M \in \mathcal{A}_0$ and hence one can move R in $\mathcal{B}_0$ by a translation parallel to its side ML. Indeed, it is easy to check that if R is in the above position, then the distance from M to the line $\{x = a\}$ is equal to cd/b and $M \in \mathcal{A}_0$ since $cd/b \le a$. $\square$

In the next proposition we characterize the rectangles that can move around the corner of an L-shaped corridor. This is done in terms of the function $f_{(a,b,c)}$, defined in (1). Its geometric meaning is the following – the value of $f_{(a,b,c)}$ at $t \in \Delta$ is equal to the distance from the point $C(a,b)$ to the line with slope $-\tan t$ that cuts a segment of length c from the first quadrant. It is clear that $f_{(a,b,c)}$ is a continuous function which is increasing in a and b, and decreasing in c.

Proposition 8. *A rectangle with side lengths c and $d, c \geq d$, belongs to the set $\mathcal{R}_{13}$ if and only if $c \leq a \vee b$ and $d \leq a \wedge b$, or $d \leq m(a,b,c)$.*

Proof. Note as above that a rectangle R with side lengths c and $d, c \geq d$, can lie in $\mathcal{A}_1$ (resp. $\mathcal{B}_3$) precisely when $d \leq a$ (resp. $d \leq b$.) Moreover, it can be moved in $\mathcal{A}_1$ (resp. $\mathcal{B}_3$) to a position where two of its sides lie on the coordinate axes. Such positions of R are called *good*. Hence we can assume that any movement of R from $\mathcal{A}_1$ into $\mathcal{B}_3$ around the corner C starts and ends at good positions and if $c \geq a \vee b$ a side of R with length c lies on Oy^+ at its initial position and on Ox^+ at the final one. (Note that that R can be moved in the corridor $\mathcal{A}_1$ so that it lies on Oy^+ on sides of lengths c and d if and only if $c^2 + d^2 \leq a^2$.)

Let $\overrightarrow{e_1} = (1,0)$ and $\overrightarrow{e_2} = (0,1)$ be the unit coordinate vectors and let $KLMN$ be the initial position of R with $M = (0,0)$ and $\overrightarrow{ML} = c\overrightarrow{e_2}$. We will say that a movement of R around C is a *rotation* (resp. *anti-rotation*) if $\overrightarrow{ML} = -c\overrightarrow{e_1}$ (resp. $\overrightarrow{ML} = c\overrightarrow{e_1}$) at its final position.

Lemma 9. *A rectangle $R \subset \mathcal{A}_1$ with side lengths c and d (not necessarily $c \geq d$) can be moved into $\mathcal{B}_3$ by a rotation around the corner C if and only if $d \leq m(a,b,c)$.*

Proof. We first prove the *only if part* of the lemma. Let $R \subset \mathcal{A}_1$ be a rectangle with side lengths c and d that can move in $\mathcal{B}_3$ by a rotation around the corner C. It is clear that $d \leq a \wedge b$ and we have to prove that if $c > a \vee b$, then $d \leq m(a,b,c)$. Let $R = KLMN$ be a rectangle in a good position with vertices as above. For any $t \in \Delta^0$ denote by $A_t \in Ox^+$ and $B_t \in Oy^+$ the points such that $C \in A_tB_t$ and $\angle OA_tB_t = t$. Consider the moment when the sides of R with length c are parallel to A_tB_t and move R by translations parallel to Oy and Ox so that $L \in Oy^+$ and $M \in Ox^+$. Consider the closer to O tangent to $k(C,d)$ which is parallel A_tB_t and denote by A and B be its intersection points with Ox^+ and Oy^+, respectively. It is clear that $|LM| \leq |AB|$ since otherwise it follows by convexity that $R \subsetneq \mathcal{C}_{13}$. Hence $d \leq f_{(a,b,c)}(t)$ for any $t \in \Delta^0$ and therefore $d \leq m(a,b,c)$.

We now prove the *if part* of the lemma. Recall that R is in a good initial position. Also, the inequality $d \leq m(a,b,c)$ implies $d \leq f_{(a,b,c)}(0) = b$ and $d \leq f_{(a,b,c)}(\pi/2) = a$. Thus if $c \leq a \vee b$, then $R \subset \mathcal{B}_3$. So, we have to show that if $c \geq a \vee b$ and $d \leq m(a,b,c)$, then we can move R to $\mathcal{B}_3$ by a rotation around C. For $t \in \Delta$ denote by R_t the rectangle with vertices $L_t = (0, c\cos t), M_t = (c\sin t, 0), N_t = (d\cos t + c\sin t, d\sin t), K_t = (d\cos t, d\sin t + c\cos t)$. The inequality $d \leq m(a,b,c)$ means that $d \leq f_{(a,b,c)}(t)$ for any $t \in \Delta$ and therefore the distance from C to the line L_tM_t is $\geq d$. Hence R moves from $\mathcal{A}_1$ into $\mathcal{B}_3$ by rotation around C. $\square$

For any $c \geq a \vee b$ we set $t_1 = \arccos(b/c), t_2 = \arcsin(a/c)$, $\widetilde{\Delta} = [t_1, t_2]$ and $\widetilde{m}(a,b,d) = \min_{\widetilde{\Delta}} f(a,b,d)$.

Lemma 10. *A rectangle $R \subset \mathcal{A}_1$ with side lengths c and d, $c \geq a \vee b$, can be moved into $\mathcal{B}_3$ by an anti-rotation around the corner C if and only if $c \leq \widetilde{m}(a,b,d)$.*

Proof. We first prove the *only if part* of the lemma. Let a rectangle $R \subset \mathcal{A}_1$ with side lengths c and d, $c \geq a \vee b$, can be moved into $\mathcal{B}_3$ by an anti-rotation around the corner C. Note that for any $t \in \Delta$ there is a moment when the angle between a side of R with length d and Ox is t. In particular, at some moment such a side of R is perpendicular to OC and therefore $c^2 \leq a^2 + b^2$ which implies that $t_1 \leq t_2$. We note that for $t \in [0, t_1]$ the rectangle R lies in the corridor $\mathcal{A}_1$, whereas for $t \in [t_2, \pi/2]$ it lies in the corridor $\mathcal{B}_3$. Hence it is enough to consider only the positions of R where $t \in \widetilde{\Delta}$. For any such t there is a moment when the sides of R with length d are parallel to the tangent to the circle $k(C; c)$ with slope $-\tan t$ and we conclude as in the proof of the *only if part* of Lemma 9 that $c \leq \widetilde{m}(a, b, d)$.

To prove the *if part* of the lemma suppose that $c \leq \widetilde{m}(a, b, d)$ and note that this inequality can be written as

$$d \leq g_{(a,b,c)}(t) = \frac{a \sin t + b \cos t - c}{\sin t \cos t}. \tag{2}$$

It is easy to check that $g_{(a,b,d)}(t)$ is the length of the segment which the tangent to the circle $k(C; d)$ and slope $-\tan t$ cuts from the first quadrant. In particular, we have from (2) that

$$d \leq g_{(a,b,c)}(t_1) = \frac{c(a - \sqrt{c^2 - b^2})}{b} \leq a$$

and

$$d \leq g_{(a,b,c)}(t_2) = \frac{c(b - \sqrt{c^2 - a^2})}{a} \leq b$$

since $c \geq a \vee b$.

Let now $R = KLMN$ be a rectangle with side lengths c and d, $c \geq a \vee b$, which is in a good position with $M = O(0,0)$ and $\overrightarrow{ML} = c\overrightarrow{e_2}$. For any $t \in \widetilde{\Delta}$ consider the rectangle $R_t = K_t L_t M_t N_t$ with vertices $K_t = (d \cos t + c \sin t, c \cos t), L_t = (c \sin t, c \cos t + d \sin t), M_t = (0, d \sin t), N_t = (d \cos t, 0)$. It is easy to check that the function $x(t) = d \cos t + c \sin t$ is increasing on the segment $[0, t_1]$ since $d \leq a$ and $x(t_1) \leq a$ since $c \leq f_{(a,b,d)}(t_1)$. Hence $K_t \in \mathcal{A}_1$ for any $t \in [0, t_1]$ which implies that $R_t \subset \mathcal{A}$ for $t \in [0, t_1]$. Similarly $R_t \subset \mathcal{B}$ for $t \in [t_2, \pi/2)$. It follows also that $R_t \subset \mathcal{C}$ for $t \in [t_1, t_2]$, since the inequality $c \leq f_{(a,b,d)}(t)$ means that the point C is at a distance at least d from the line $M_t N_t$. □

Remark 11. *The above proof easily implies that Lemma 10 remains also true if $c < b$ or/and $c < a$, replacing t_1 by 0 or/and t_2 by $\pi/2$ and adding the condition $d \leq a \wedge b$.*

We are now ready to prove Proposition 8. It follows from Lemma 1 and Lemma 2 that it is enough to prove that if $c > a \vee b$ and a rectangle in $\mathcal{A}_1$ with side lengths c and $d, c \geq d$, can be moved into $\mathcal{B}_3$ by anti-rotation around the corner C, then the same can be done by a rotation around C. In other words, if $c \leq \widetilde{m}(a, b, d)$, then $d \leq m(a, b, c)$. Writing the first inequality in the form $d \leq \min_{\widetilde{\Delta}} g_{(a,b,c)}(t)$, where $g_{(a,b,c)}$ is the function defined by (2), we have to prove that

$$\min_{\Delta} f_{(a,b,c)}(t) \geq \min_{\widetilde{\Delta}} g_{(a,b,c)}(t). \tag{3}$$

To do this consider the function

$$h(t) = f_{(a,b,c)}(\pi/2 - t) = a\cos t + b\sin t - c\sin t\cos t,\ \ t \in \Delta.$$

It is clear that $\min_\Delta f_{(a,b,c)}(t) = \min_\Delta h(t)$. In addition, it follows easily from $c \ge a \vee b$ that $h(t) \ge g(t)$ for any $t \in \Delta$. On the other hand the function $f_{(a,b,c)}(t)$, and hence $h(t)$, has a unique minimum in the interval Δ (see **(A)** in Appendix) and let that of $h(t)$ is attained at $t_* \in \Delta$. Direct computations show that

$$h'(t_1) = \frac{\sqrt{c^2-b^2}(\sqrt{c^2-b^2}-a)}{c}\ ,\ \ h'(t_2) = \frac{\sqrt{c^2-a^2}(b-\sqrt{c^2-a^2})}{c}.$$

Since $c^2 \le a^2 + b^2$ (see the proof of Lemma 2) we conclude that $h'(t_1) \le 0 \le h'(t_2)$, i.e. $t_* \in \widetilde{\Delta}$. Then

$$\min_\Delta f_{(a,b,c)}(t) = \min_\Delta h(t) = h(t_*) \ge g(t_*) \ge \min_{\widetilde{\Delta}} g(t)$$

and inequality (3) is proved. □

Corollary 12. *Let $a = b = 1$ and $R \subset \mathcal{A}_1$ be a rectangle with side lengths c and d. Then R can be moved into $\mathcal{B}_3$ around the corner C by:*

(i) a rotation if and only if $c \le 2(\sqrt{2}-1)$ and $d \le 1$, or $2(\sqrt{2}-1) \le c \le 2\sqrt{2}$ and $d \le \sqrt{2} - c/2$;

(ii) an anti-rotation if and only if $c \le \sqrt{2} - 1/2$ and $d \le 1$, or $\sqrt{2} - 1/2 \le c \le \sqrt{6} - \sqrt{2}$ and $d \le 2\sqrt{2} - 2c$, or $\sqrt{6} - \sqrt{2} \le c \le \sqrt{2}$ and $d \le c(1 - \sqrt{c^2-1})$.

Proof. (i) The derivative of the function $f_{(1,1,c)}$ is given by

$$f'_{(1,1,c)}(t) = (\cos t - \sin t)(1 - c(\sin t + \cos t)).$$

Note that the equation

$$1 - c(\sin t + \cos t) = 0$$

has two distinct roots in Δ (with sum $\pi/2$) for $c \in (\sqrt{2}/2, 1]$, and it has a double root $\pi/4$ for $c = \sqrt{2}/2$; otherwise, it has no roots in Δ. Hence $m(1,1,c) = f_{(1,1,c)}(0) = f_{(1,1,c)}(1) = 1$ for $c < \sqrt{2}/2$, $m(1,1,c) = f_{(1,1,c)}(\pi/4) = \sqrt{2} - c/2$ for $c > 1$, and $m(1,1,c) = \min\{1, \sqrt{2} - c/2\}$ for $c \in [\sqrt{2}/2, 1]$. Therefore (i) follows from Lemma 9.

(ii) To prove this statement we will use the function $g_{(1,1,c)}$ defined in (2). Its derivative is given by

$$g'_{(1,1,c)}(t) = \frac{(\sin t - \cos t)(1 + \sin t\cos t - c(\sin t + \cos t))}{\sin^2 t\cos^2 t}.$$

Note that the equation

$$1 + \sin t\cos t - c(\sin t + \cos t) = 0$$

has two distinct roots in Δ^0 (with sum $\pi/2$) for $c \in (1, \frac{3\sqrt{2}}{4})$ and a double root $\pi/4$ for $c = \frac{3\sqrt{2}}{4}$; otherwise, it has no roots in Δ^0. Hence $\min_{\Delta^0} g_{(1,1,c)} = g_{(1,1,c)}(\pi/4) = 2\sqrt{2} - 2c$ for $c \le 1$, $\min_{[t_1,t_2]} g_{(1,1,c)} = g_{(1,1,c)}(t_1) = g_{(1,1,c)}(t_2) = c(1 - \sqrt{c^2-1})$ for $c > \sqrt{6} - \sqrt{2}$, and $\min_{[t_1,t_2]} g_{(1,1,c)} = \min\{2\sqrt{2}-2c, c(1-\sqrt{c^2-1})\}$ for $c \in [1, \frac{3\sqrt{2}}{4}]$. Note that for $c \geq 1$,

$$2\sqrt{2}-2c \leq c(1-\sqrt{c^2-1}) \Leftrightarrow c^2(c^2-1) \leq (3c-2\sqrt{2})^2$$
$$\Leftrightarrow (c-\sqrt{2})^2(c-\sqrt{6}+\sqrt{2})(c+\sqrt{6}+\sqrt{2}) \leq 0.$$

Therefore (ii) follows from the proof of Lemma 10 and Remark 11. □

Remark 13. *Let $a=b=1$ and $R \subset \mathcal{A}_1$ be a rectangle with side lengths c and d. As we know from the proof of Proposition 8 (and also from the above corollary), if $c \geq 1$ and R can be moved into the corridor $\mathcal{B}_3$ by an anti-rotation, then the same can be done by a rotation. Note, however, that for $c<1$ this is not always true. For example, if $d=1$ and $c \in (2(\sqrt{2}-1), \sqrt{2}-1/2]$ it follows from Corollary 12 that R can be moved into the corridor $\mathcal{B}_3$ by an anti-rotation, but this cannot be done by a rotation.*

3. Proof of Theorem 1

The equalities for the sets $\mathcal{R}_{ij}$ follow from their characterizations in statements (i)–(iv) which we prove below.

3.1. **Proof of the *if part* of Theorem 1.** We will prove the *if part* of each of the statements (i)-(iv).

(i) It follows from the proof of the *if part* of Proposition 7 since the movement for the rectangles in $\mathcal{R}_{00}$ used there works for the rectangles in $\mathcal{R}_{01}$ and $\mathcal{R}_{02}$ as well.

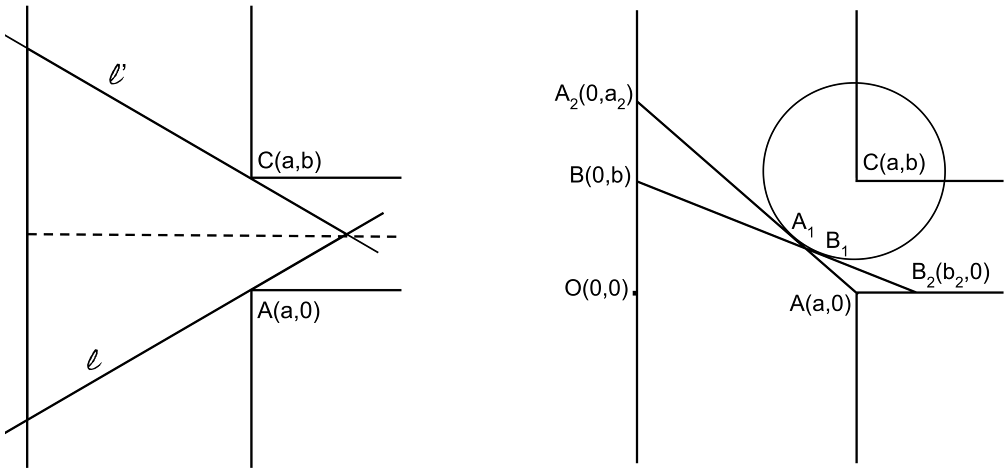
(ii) Let $h \leq d \leq a \wedge b$ and $cd \leq ab$. Then we move the rectangle $R = KLMN$ as in the proof of Proposition 7 to a position where $K = A, L \in \mathcal{B}_3$ and $M, N \in \mathcal{A}_0$. We can assume that $N \in Oy^+$ since otherwise we replace the "wall" Oy of the corridor $\mathcal{A}_0$ with the line $\{x = a - a'\}$, where a' is the distance from N to the line $\{x = a\}$. The new corridor has width $a' < a$ and the given conditions are satisfied since $d \geq h > h'$ and we know from the proof of Proposition 7 that $cd \leq a'b$. Consider the tangent to $k(C; d)$ from $A(a, 0)$ and use the same notations as in the proof of the *only if part* of (ii) below (Fig. 3b). Then

$$d = \frac{ab}{\sqrt{a^2+a_2^2}}\ ,\ h = \frac{ab}{\sqrt{a^2+b^2}}$$

which imply that $d > h \Leftrightarrow a_2 < b$. We have also that $c \leq ab/d = |AA_2|$. Using Remark **(B)** in Appendix we conclude that this inequality holds for the tangents to $k(C; d)$ from the points on Ox with abscissa $\geq a$. Now it follows from the proof of Lemma 9 that one can move R in $\mathcal{B}_3$ by a rotation around the corner C.

Finally, if $d \leq h \wedge m(a, b, c)$, then the desired movement of R in B_3 is the same as that in the proof of the *if part* of Lemma 10 .

(iii) Since $\mathcal{R}_{12} \subset \mathcal{R}_{10}$ we have to prove that a rectangle R in $\mathcal{A}_1$ with side lengths satisfying one of the three conditions can move in $\mathcal{B}_2$. If $d \leq h$, then it follows from Remark **(B)** in Appendix that $[AA_2]$ is the shortest segment with ends on Oy and $[OA]$, respectively which is tangent to the circle $k(C; d)$. This segment has length ab/d and it follows from Proposition 7 and Proposition 8 that in all three cases the rectangle R can be moved from $\mathcal{A}_1$ into $\mathcal{B}_2$. Note that to apply Proposition 7 we need



(A) (B)

FIGURE 3. The corridor $\mathcal{C}_{03}$.

to show that the condition $d \leq m(a,b,c)$ implies the inequality $cd \leq ab$. To see this set $\varphi = \arccos(a/l)$. Then

$$cd \leq cf_{(a,b,c)}(\varphi) = abc(2/l - c/l^2) \leq ab \tag{4}$$

with equality only if $c = l, d = h$.

(iv) It follows from the proof of the *if part* of Proposition 8 since the movement for the rectangles in $\mathcal{R}_{13}$ used there is not affected by the upper wall of the corridor $\mathcal{B}_3$ and it works for the rectangles in $\mathcal{R}_{11}$ as well.

3.2. **Proof of the *only if part* of Theorem 1.** We will prove the *only if part* of each of the statements (i)-(iv).

(i) It follows from Proposition 7 since $\mathcal{R}_{01} \cup \mathcal{R}_{02} \subset \mathcal{R}_{00}$.

(ii) Let $R \subset \mathcal{A}_0$ be a rectangle with side lengths c and d, $c \geq d$, which moves in $\mathcal{B}_3$, around the corner C. If $d > h$ the *only if part* follows from Proposition 8. Let now $d < h$. We can assume that the initial position of R is such that its sides with length c are parallel to Oy, while at its final position they are parallel to Ox. Let l be the tangent to the circle $k(C,d)$ for which the minimum $n(a,b,d)$ of $g_{(a,b,d)}$ is attained and l' be the line which is symmetric to l with respect to the line $y = b/2$ (Fig. 3a). Then at some moment the sides of R with length c are parallel either to l, or to l' and by symmetry, we can assume that these sides are parallel l. Consider the tangents from A and B to the circle $k(C,d)$ such that they both intersect Ox^+ and Oy^+. Denote by A_1 and B_1 their tangent points to $k(C,d)$, and by $A_2 = (0,a_2)$ and $B_2 = (b_2,0)$ their intersection points with Oy^+ and Ox^+, respectively (Fig. 3b). Since $d < h$ it follows that $a_2 > b$ and $b_2 > a$ which shows that the line l intersects the segments $[B, A_2]$ and $[A, B_2]$. Now it is easy to see that at the moment when the sides of R with length c are parallel l we have $c \le n(a,b,d)$ whatever is the position of R. Hence $d \le m(a,b,c)$ and the *only if part* for $d < h$ is also proved.

(iii) It follows from Proposition 7 that it is enough to prove that if $h < d < a \wedge b$, $c > a \vee b$ and $R \in \mathcal{R}_{10}$, then $d \le m(a,b,c)$.

Let ℓ be the line through O which is parallel to the sides of R with length c and let $\varphi \in [0, \pi/2]$ be the angle between ℓ and Ox. As in the proof of Proposition 7 we can assume that $\varphi = \pi/2$ in the initial position of R. Consider the last moment until at least three of its vertices lie in $\mathcal{A}_1$. Having in mind that $c > a \vee b$, it follows that at that moment $R = KLMN$, where $|KL| = d$ and $K \in \mathcal{D}$, $L \in \mathcal{B}_1 \setminus \mathcal{D}$, $M, N \in \mathcal{A}_1 \setminus \mathcal{D}$. Let y_K and y_L be the ordinates of K and L. It is clear that $y_K \neq y_L$. If $y_K > y_L$, then $R \subset \mathcal{B}_3$ and the inequality $d \le m(a,b,c)$ follows from Proposition 8. Let now $y_K < y_L$. Then one can easily see that $\varphi \le \psi$, where ψ is the angle corresponding to the tangent to the circle $k(C; d)$ through A. Since $d > h$, it follows from Remark **(B)** in Appendix that $\psi < \psi_0$, where ψ_0 is the angle corresponding to the tangent to $k(C; d)$ for which the minimum $n(a,b,d)$ of the function $g_{(a,b,d)}$ is attained. It follows by continuity that at some moment the line ℓ is parallel to that tangent. For such a position of R it is clear that $c \le n(a,b,d)$ which is equivalent to $d \le m(a,b,c)$.

(iv) It follows from Proposition 8 since $\mathcal{R}_{13} \subset \mathcal{R}_{11}$. □

4. Proof of Theorem 4

Let P be a rectangular parallelepiped that can move around the corner of the corridor S_{ij}. We will show that $V_P \le abc$. To do this consider the last moment until at least six vertices of P lie in $\mathcal{A}_i \times (0, c)$. Set $\mathcal{D}' = \overline{\mathcal{D}} \times [0, c]$. Then at least three vertices K, L, M of a face of P lie in $\mathcal{B}_j \times (0, c) \cup \mathcal{D}'$ with $M \in \mathcal{D}'$ and the corresponding vertices K_1, L_1, M_1 of its opposite face lie in $\overline{\mathcal{A}_i} \times [0, c]$. It is clear that the edges KK_1 and LL_1 meet $\mathcal{D}'$ at some points K_2 and L_2. Let the line MM_1 meet the plane $\{x = 0\}$ at a point M' and let θ be the angle between this plane and the plane (KLM). Then

$$2S_{\triangle KLM} = 2S_{\triangle K_2 L_2 M} \cos\theta \le bc\cos\theta,\ \ MM_1 \le MM' = \frac{a}{\cos\theta},$$

and we get $V_P \le abc$.

If $V_P = abc$, then $2S_{\triangle K_2L_2M} = bc$, i.e. two of the vertices of $\triangle K_2L_2M$ are adjacent vertices of $\mathcal{D}'$, and the third one is lying on its opposite side. Then it is easy to see that $P = R \times (0, c)$, where $R \in \mathcal{R}_{ij}$. (In fact $M_1 = M_1'$ and $\mathcal{D}' = P \cap \{x = 0\}$.) Now Theorem 4 follows by Lemma 6 which we prove below.

Proof of Lemma 6. Let $\mathcal{C} \subset P$ be a right circular cylinder whose basses circles lie on α and β and have diameter $r > 0$. Suppose that at some moment of the movement of P in the layer between α and β it, and hence $\mathcal{C}$, is inclined to these planes. Consider the intersection of $\mathcal{C}$ with the plane through the centers of its basses circles and orthogonal to α and β. It is a rectangle $R = KLMN$ with $|KL| = |MN| = c$(the distance between α and β) and $|NK| = |ML| = r$. Let $NK \cap \alpha = K_0, MN \cap \alpha = N_0$ and $\angle N_0K_0N = \varphi_0$. We can assume that $\mathcal{C}$ is inclined to α and β such that $\varphi_0 \in (0, \pi/2)$.Then

$$r \le |N_0N| \cot\varphi_0 \le \left(\frac{c}{\cos\varphi_0} - c\right)\cot\varphi_0 = c\tan\frac{\varphi_0}{2}. \tag{5}$$

and it follows by continuity that this inequality holds for all $\varphi \in (0, \varphi_0)$. Now letting $\varphi \to 0+$ in (5) we get $r \le 0$, a contradiction. □

5. Appendix

In this section we prove some technical facts that are used in the proofs of the results stated in Section 1.

A. Note that in some particular cases the minimum $m(a, b, c)$ of the function $f_{(a,b,c)}$ can be found explicitly. For example, it follows by Remark **(B)** below that $m(a, b, l) = h$ and $m(a, b, c) = 0$ for $c \ge (a^{2/3} + b^{2/3})^{3/2}$. We also know from Corollary 12 that $m(a, a, c) = a\sqrt{2} - c/2$ for $c \ge a$.

In the general case $m(a, b, c)$ is a rational function of the largest root of a quartic equation and hence can be found in radicals with respect to a, b, c. Indeed, the derivative of $f_{(a,b,c)}$ is given by

$$f'_{(a,b,c)}(t) = a\cos t - b\sin t - c\cos 2t$$

and setting $x = \tan t$ for $|t| < \pi/2$ we get

$$p(x) := (x^2 + 1)f'_{(a,b,c)}(\arctan x) = (a - bx)\sqrt{x^2 + 1} + c(x^2 - 1).$$

We set also

$$q(x) = (a - bx)\sqrt{x^2 + 1} + c(1 - x^2)$$

and note that $r = pq$ is a polynomial of degree 4. One checks easily that $p(\pm\infty) = +\infty$, $p(0) < 0$, $q(\pm\infty) = -\infty$ and $q(0) > 0$. Therefore, each of the functions p and q has exactly one positive and one negative zero and these are all four zeros of r. Denote by x_1 and y_1 the positive zeros of p and q, respectively. If $a = b$, then $x_1 = y_1 = 1$. If, for example $a < b$, then $p(1) = q(1) < 0$. Hence $y_1 < 1 < x_1$ and x_1 is the largest zero of r. Therefore we can find x_1 in radicals with respect to a, b, c, and do the same for $m(a, b, c) = f(\arctan x_1)$.

B. For fixed $a, b > 0$ and $0 \le d \le a \wedge b$ consider $n(a, b, d) := \inf_{\Delta^o} g_{(a,b,d)}$, where

$$g_{(a,b,d)}(t) := \frac{a\sin t + b\cos t - d}{\sin t\cos t},\ t \in \Delta^o.$$

Recall that $g_{(a,b,d)}(t)$ is the length of the segment which the tangent to the circle $k(C; d)$ at the point $(a - d\sin t, b - d\cos t)$ cuts from the first quadrant. Hence $n(a, b, d)$ is the length of the shortest such segment.

Straightforward computations show that

$$g'_{(a,b,d)}(t) = \frac{a\sin^3 t - b\cos^3 t + d\cos 2t}{\sin^2 t\cos^2 t}$$

and

$$g''_{(a,b,d)}(t)\sin^3 t\cos^3 t = b(\cos^3 t + \cos^5 t) + a(\sin^3 t + \sin^5 t) - d(1 + \cos^2 2t) \ge$$
$$d((1 - \sin t)^2\sin^3 t + (1 - \cos t)^2\cos^3 t) \ge 0.$$

Hence $g_{(a,b,d)}$ is a convex function. Since $g'(0+) = -\infty$ for $d < b$, $g'(0+) = -d/2$ for $d = b$, $g'(\pi/2-) = +\infty$ for $d < a$ and $g'(\pi/2-) = d/2$ for $d = a$ it follows that g' has a unique zero $t_0 \in \Delta^o$ and $n(a,b,d) = g_{(a,b,d)}(t_0)$.

Set $x = \tan t$. Then the equation $g'_{(a,b,d)}(t) = 0$ takes the form $u(x) = 0$, $x > 0$, where

$$u(x) = ax^3 - b - d(x^2-1)\sqrt{x^2+1}.$$

We know that u has a unique positive zero $x_0 = \tan t_0$ and this zero is simple since $g''_{(a,b,d)} > 0$. As in **(A)** we set

$$v(x) = ax^3 - b + d(x^2-1)\sqrt{x^2+1}$$

and note that uv is a polynomial of degree 6. One easily checks that $v'(x) > 0$ for $x > 0$, $v(0) < 0$ and $v(+\infty) = +\infty$ which show that v has a unique positive zero x_1 and this zero is simple. Hence x_0 and x_1 are the only positive zeros of uv. If $a = b$, then $x_0 = x_1 = 1$, and if, for example, $b < a$, then $0 < x_0 < \sqrt[3]{b/a} < x_1 < 1$ (compare with [6]).

It is clear that $n(a,b,d)$ is a continuous and strictly decreasing function in $d \in [0, a \wedge b]$. Let $d \neq 0$ and $\varphi = \arccos(a/l)$. Then

$$g_{(a,b,d)}(\varphi) = ab\left(\frac{2}{l} - \frac{d}{l^2}\right) \leq \frac{ab}{d}$$

which implies that

$$n(a,b,d) \leq \frac{ab}{d} \tag{6}$$

with equality if and only if $d = h$. Hence $n(a,b,h) = l$ and $m(a,b,l) = h$.

Now consider the case $d = 0$ which corresponds to the *leader problem* [3] .It follows easily by the above results that in this case $\tan t_0 = (b/a)^3$ and $n(a,b,0) = (a^{2/3} + b^{2/3})^{3/2}$. This together with Theorem 1 implies Corollary 2.

Finally, if $a = b$, then $t_0 = \pi/4$ and $n(a,a,d) = 2\sqrt{2}a - 2d$. which we know from Corollary 12.

Note that the above three particular cases are related to [6, Questions 2 and 6, p. 200].

C. The condition $d \leq m(a,b,c)$ for rotation around the corner C (see e.g. Lemma 9) can be written as $a \geq k(c,d,b) := \sup_{\Delta'} h_{(c,d,b)}$, where $\Delta' = (0, \pi/2]$ and

$$h_{(c,d,b)}(t) = c\cos t + \frac{d}{\sin t} - b\cot t.$$

It is clear that $d \leq k(c,b,d)$, and that $k(c,d,b) < +\infty$ is equivalent to $d \leq b$.

We have

$$w(t) := h'_{(c,d,b)}(t)\sin^2 t = b - c\sin^3 t - d\cos t,\ 2w'(t) = \sin t(2d - 3c\sin 2t).$$

If $2d < 3c$, then w has two zeros $t_1 < t_2 \in \Delta^0, t_1 + t_2 = \pi/2$, w is strictly increasing in the intervals $[0, t_1]$ and $[t_2, \pi/2]$, and it is strictly decreasing in the interval $[t_1, t_2]$. In particular (c.f. [5]) if $b \leq c$, then w has a unique zero $t_* \in \Delta^0$ and $t_1 < t'_1 \leq t_* \leq t'_2 < t_2$, where $w(t'_1) = w(0)$ and $w(t'_2) = w(\pi/2)$. Therefore $k(c,d,b) = h_{(c,d,b)}(t_*) \geq d$. (Note that similarly to **(B)**, $\arctan t_*$ is a zero of a six degree polynomial.)

Having in mind the inequality $k(c, d, b) \geq d$ it is natural to ask when the equality is attained. In other words the question is when given $b \geq d$ and c, the least a such that $m(a, b, c) \geq d$, is equal d. We will show that this is so if $c \leq (\sqrt{2} - 1/2)d$. To prove this it is enough to check that $m(a, b, c) = d$ if $a = b = d$ and $c = (\sqrt{2} - 1/2)d$ which follows from **(A)** (as well as from **(B)**, interchanging c and d). Note that the above constraint for c is the optimal one.

One can prove in the same way that given $a \leq b$ and $c \leq (\sqrt{2} - 1/2)a$, the largest d for which $m(a, b, c) \geq d$ is equal to a.

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