

HOW REGULAR IS THE EVOLUTE OF A PLANE CURVE?

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ABSTRACT. We study the relationship between the smoothness of a plane curve and that of its evolute, especially in the cases where the parent curve is no more two or three times continuously differentiable, and exhibit the same kind of apparent improvement in regularity: in the generic local situation, the evolute has one order of regularity less than the parent curve.

1. INTRODUCTION

1.1. Motivations and outline. The notions of osculating circle and center of curvature of a smooth enough plane curve are classical, going back at least to Leibnitz. The locus of all the centers of curvature of a given curve is called its *evolute* and turns out to be the geometric envelope of the family of normal lines to the curve. The inverse operation—finding an *involute*, a curve which the given curve is the evolute of—has even found some applications to the design of some mechanisms, see for instance [11]. This has other beautiful geometrice properties, some less well-known, such as the Tait-Kneser Theorem [9]: if the radius of curvature varies in a monotone way, then the osculating circles are nested. Observe that the converse of this is immediate. Most of this can be found in elementary treatises about differential geometry or advanced calculus; a good succinct contemporary account can be found in [3], which singles out the Tait-Kneser Theorem as worthy of more attention.

However, the examples for which evolutes have been studied are mostly infinitely differentiable (cycloids), or even algebraic curves (conic sections), see the Wikipedia page [10] or e.g. [8]. When the curves are bounded, their curvature must admit at least four local extrema (Four Vertex Theorem [6]), which generate cusps in the evolute; the interesting generalization given in [2] (and subsequent papers) studies such singularities, but always in the context of the parent curve being infinitely differentiable.

In contrast to this, we are interested in the least possible amount of smoothness to be required from the parent curve in order to get a certain smoothness for the evolute, and conversely. At a first glance, it would seem that since the curvature depends on second derivatives and the evolute is expressed in terms of it, it should have two orders of regularity less than the parent curve. The result however is that the evolute has only one order of regularity less than the parent curve, provided that the curvature of that curve has non-vanishing derivative, see Theorem 1. This “bootstrap” phenomenon, an apparent regularity gain due to geometric constraints, reminds us of what happens when looking at distance functions and was pointed out in [5], who attribute the initial observation to the appendix of [4], see [7] for recent refinements.

Our results are local, since globally an evolute can have double points; we provide an example to that effect.

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Much of the present paper may be known, but not always easy to find in current references, and we believe we have a relatively concise exposition of the results, and proofs which do not require many prerequisites.

1.2. Definitions and notations. A plane curve is a continuous map γ from an interval $(a; b)$ to the plane $\mathbb{R}^2$. For $k \in \mathbb{Z}_+$, $\alpha \in (0, 1)$, we say the curve is of class $\mathcal{C}^k$ (resp. $\mathcal{C}^{k,\alpha}$) if its components are. In order to unify notation, we will write $\mathcal{C}^{k,0}$ for $\mathcal{C}^k$. Recall that a function is of class $\mathcal{C}^\infty$ if it is of class $\mathcal{C}^k$ for any k . We routinely use the fact that if $f, g \in \mathcal{C}^{k,\alpha}$, with $k \geq 1$, then $f \circ g \in \mathcal{C}^{k,\alpha}$ (this of course fails for $k = 0, \alpha < 1$).

For $k \geq 1$, we say that the curve is $\mathcal{C}^{k,\alpha}$ -regular if in addition, for all $t \in (a; b)$, $\gamma'(t) \neq (0, 0)$. In this case, there exists an open interval I containing t such that $\gamma|_I$ embeds I into $\mathbb{R}^2$ as a one-dimensional submanifold of the same regularity as γ . Note that this does not imply that γ is globally one-to-one, only over a small enough interval.

For a curve of class $\mathcal{C}^1$ and $t_0 \in (a; b)$, the *arc length* is $s_\gamma(t) := \int_{t_0}^t |\gamma'(u)| du$. Changing the point t_0 changes $s_\gamma(t)$ by an additive constant, and we will have to do that on occasion. When γ is, in addition, regular, s_γ is an increasing diffeomorphism from $(a; b)$ to $(s_\gamma(a), s_\gamma(b))$, of at least the same smoothness class as γ ; the arclength parametrization therefore provides the best possible smoothness for the curve among the regular parametrizations. We will consider $\gamma \circ s_\gamma^{-1}$ as the “same curve parametrized by arc length”, but will need to distinguish it in notation.

When γ is parametrized by arc length, $|\gamma'(s)| = 1$ for any $s \in (a; b)$. There is a lifting to $\mathbb{R}$ of the map $\gamma' : (a; b) \rightarrow S^1$, so we denote $\gamma'(s) = (\cos \varphi(s), \sin \varphi(s))$ where $\varphi \in \mathcal{C}^{k-1,\alpha}(a; b)$ when γ is of class $\mathcal{C}^{k,\alpha}$. Let $\nu(s) := (-\sin \varphi(s), \cos \varphi(s))$ (unit normal vector) be $\gamma'(s)$ rotated by $+\frac{\pi}{2}$.

When $k \geq 2$, $\kappa(s) := \varphi'(s)$ is called the (*signed*) *curvature* of γ . Reducing the interval if needed, we make the standing assumption that for $s \in (a; b)$, $\kappa(s) \neq 0$. Then we can define $R(s) := 1/\kappa(s)$ is the (*signed*) *radius of curvature* of γ . The well-known (plane) Frenet-Serret formulae immediately follow:

$$\gamma''(s) = \frac{1}{R(s)}\nu(s), \quad \nu'(s) = -\frac{1}{R(s)}\gamma'(s).$$

The *evolute* of the curve γ is given by the map

$$\tilde{\gamma}(s) := \gamma(s) + R(s)\nu(s).$$

When we parametrize the evolute by its own arc length $\tilde{s}$, we write $\tilde{s} \mapsto \tilde{\gamma}_1(\tilde{s}) = \tilde{\gamma}(s)$.

1.3. Main results.

Theorem 1. *Let $\gamma : (a; b) \rightarrow \mathbb{R}^2$ be a $\mathcal{C}^2$ -regular curve such that $\kappa(t) \neq 0$ for all $t \in (a; b)$, and denote its evolute by $\tilde{\gamma}$. Then*

- (i) $\tilde{\gamma}$ is $\mathcal{C}^1$ -regular if and only if R (or equivalently κ) is strictly monotone on $(a; b)$.
- (ii) For $k \geq 3$, $0 \leq \alpha \leq 1$: γ is $\mathcal{C}^{k,\alpha}$ -regular and $R'(s) \neq 0$, if and only if $\tilde{\gamma}$ is $\mathcal{C}^{k-1,\alpha}$ -regular.

Note that nonvanishing of the derivative of the radius of curvature implies its monotonicity, since we are working on an interval.

The fact that monotonicity of the radius of curvature is necessary and sufficient for smoothness of the evolute is well known, and mentioned for instance in [1, p. 308].

2. PROOFS

Lemma 2. *Let γ be a $\mathcal{C}^2$ -regular plane curve parametrized over (a, b) by arc length s , and $a < s_1 < s_2 < b$. Let $\tilde{\gamma}$ be its evolute, parametrized by the same $s \in (a, b)$. Then*

$$(1) \quad \tilde{\gamma}(s_2) - \tilde{\gamma}(s_1) = \int_{s_1}^{s_2} \frac{R(s) - R(s_2)}{R(s)} \gamma'(s) ds + (R(s_2) - R(s_1))\nu(s_1).$$

Proof. Observe first that the Frenet-Serret formulae (and the continuity of R) imply

$$\nu(s_2) - \nu(s_1) = - \int_{s_1}^{s_2} \frac{1}{R(s)} \gamma'(s) ds = - \frac{1}{R(s_2)} (\gamma(s_2) - \gamma(s_1)) + \int_{s_1}^{s_2} \left(\frac{1}{R(s_2)} - \frac{1}{R(s)} \right) \gamma'(s) ds.$$

Then notice that

$$\tilde{\gamma}(s_2) - \tilde{\gamma}(s_1) = \gamma(s_2) - \gamma(s_1) + R(s_2)(\nu(s_2) - \nu(s_1)) + (R(s_2) - R(s_1))\nu(s_1).$$

□

Proof of Theorem 1, (i).

Suppose that R is strictly monotone on $(a; b)$, we may assume that it is increasing. Then, since $|\gamma'(s)| = 1$ and $0 \leq R(s_2) - R(s) \leq R(s_2) - R(s_1)$ for $s_1 \leq s \leq s_2$,

$$\left| \int_{s_1}^{s_2} \frac{R(s) - R(s_2)}{R(s)} \gamma'(s) ds \right| \leq |R(s_2) - R(s_1)| \int_{s_1}^{s_2} \frac{1}{|R(s)|} ds \leq |s_2 - s_1| \frac{|R(s_2) - R(s_1)|}{\min_{[s_1, s_2]} |R|}.$$

So (1) implies

$$(2) \quad \frac{\tilde{\gamma}(s_2) - \tilde{\gamma}(s_1)}{R(s_2) - R(s_1)} = \nu(s_1) + O(|s_2 - s_1|).$$

This implies that the curve admits a tangent vector, parallel to the unit vector $\nu(s)$, and that $s \mapsto R(s)$, which is a homeomorphism of intervals (by strict monotonicity) provides a (continuous) parametrization by arclength. Let $\tilde{s} := R(s)$, $\tilde{\gamma}_1(\tilde{s}) = \tilde{\gamma}(R^{-1}(\tilde{s}))$ and $\tilde{\gamma}'_1(\tilde{s}) = \nu(R^{-1}(\tilde{s}))$, which is a continuous non-vanishing vector function of $\tilde{s}$, so $\tilde{\gamma}_1$, and therefore $\tilde{\gamma}$, is $\mathcal{C}^1$ -regular.

Conversely, assume that near a point s_1 , R is not strictly monotone.

Exchanging the roles of s_1 and s_2 in (1), we have

$$\tilde{\gamma}(s_2) - \tilde{\gamma}(s_1) = \int_{s_1}^{s_2} \frac{R(s) - R(s_1)}{R(s)} \gamma'(s) ds + (R(s_2) - R(s_1))\nu(s_2).$$

First notice that we can find values of $s_2 \geq s_1$ arbitrarily close to s_1 such that $|R(s_2) - R(s_1)| = \max_{s \in [s_1, s_2]} |R(s) - R(s_1)|$, and likewise for $s_2 \leq s_1$. The argument above, applied to sequences of such values $s_{2,n} \rightarrow s_1$, leads to

$$\frac{\tilde{\gamma}(s_{2,n}) - \tilde{\gamma}(s_1)}{R(s_{2,n}) - R(s_1)} = \nu(s_{2,n}) + O(|s_{2,n} - s_1|),$$

and since $\nu(s_{2,n}) \rightarrow \nu(s_1)$, in all cases $\nu(s_1)$ belongs to the tangent cone of $\tilde{\gamma}$ at $\tilde{\gamma}(s_1)$.

There are three cases.

Case 1.

If R admits a strict local extremum at s_1 , assume without loss of generality that it is a local maximum. With $\langle \cdot, \cdot \rangle$ standing for the Euclidean inner product in the plane,

$$(3) \quad \langle \tilde{\gamma}(s_2) - \tilde{\gamma}(s_1), \nu(s_1) \rangle = (R(s_2) - R(s_1)) + \int_{s_1}^{s_2} \frac{R(s) - R(s_2)}{R(s)} \langle \gamma'(s), \nu(s_1) \rangle ds,$$

so for any $\varepsilon > 0$, if $s_2 = s_1 + h$ with h chosen as above, $|R(s) - R(s_2)| \leq |R(s_1) - R(s_2)|$ for $s \in [s_1, s_2]$ and $\langle \gamma'(s), \nu(s_1) \rangle = \langle (\gamma'(s) - \gamma'(s_1)), \nu(s_1) \rangle = O(|s - s_1|)$, so the integral term is bounded in modulus by $C|s_1 - s_2|^2 |R(s_1) - R(s_2)|$ and $\langle \tilde{\gamma}(s_2) - \tilde{\gamma}(s_1), \nu(s_1) \rangle > 0$. For values $h < 0$ close enough to 0, the same argument again yields $\langle \tilde{\gamma}(s_2) - \tilde{\gamma}(s_1), \nu(s_1) \rangle > 0$, so $\tilde{\gamma}$ cannot be regular at s_1 .

Case 2.

If R admits a non-strict local extremum at s_1 , there must be values of h arbitrarily close to 0 such that $R(s_1 + h) = R(s_1)$. So if $\tilde{\gamma}$ was differentiable at s_1 , we would have $\tilde{\gamma}'(s_1) = 0$, and this would be true for any (monotone) reparametrization of the curve, so it cannot be regular.

Case 3.

Finally if R does not admits a strict local extremum at s_1 , and is not strictly monotone in any neighborhood of s_1 , there must be a sequence of values $h_n \rightarrow 0$ such that R admits a local extremum at $s_1 + h_n$, so $\tilde{\gamma}$ cannot be $\mathcal{C}^1$ -regular at s_1 .

Proof of Theorem 1, (ii).

Direct part:

Since $R(s) = \left(\langle \gamma''(s), \frac{\gamma'(s)}{|\gamma'(s)|} \rangle \right)^{-1}$, it is immediate that R and $\tilde{\gamma}$ are of class $\mathcal{C}^{k-2,\alpha}$. The hypothesis about the derivative of R means that R is a $\mathcal{C}^{k-2,\alpha}$ diffeomorphism onto its image interval. When we reparametrize $\tilde{\gamma}$ by R , $\tilde{\gamma}'_1(\tilde{s}) = \nu(R^{-1}(\tilde{s}))$. The vector ν is obtained by a rotation of $+\pi/2$ from $\gamma'(s)$ and so $s \mapsto \nu(s)$ is of class $\mathcal{C}^{k-1,\alpha}$. Since R^{-1} is of class $\mathcal{C}^{k-2,\alpha}$ and $k-2 \geq 1$, $\tilde{\gamma}'_1$ is a $\mathcal{C}^{k-2,\alpha}$ non-vanishing vector function of $\tilde{s}$, so $\tilde{\gamma}_1$ is $\mathcal{C}^{k-1,\alpha}$ -regular.

Converse:

Now assume that $\tilde{\gamma}$ is $\mathcal{C}^{k-1,\alpha}$ -regular.

It is a classical fact (see e.g. [1] or [11]) that if we denote by $\tilde{s} \mapsto \tilde{\gamma}_1(\tilde{s})$ the arclength parametrization of the evolute, then there exists some $c \in \mathbb{R}$, necessarily outside of the source interval of that parametrization since we always assume that the original curve γ is $\mathcal{C}^2$ -regular, such that a parametrization of γ is given by

$$(4) \quad \gamma_1(\tilde{s}) = \tilde{\gamma}_1(\tilde{s}) + (c - \tilde{s})\tilde{\gamma}'_1(\tilde{s}),$$

which shows that under our hypothesis γ_1 is of class $\mathcal{C}^{k-2,\alpha}$ at least, with $k-2 \geq 1$, and so is the arclength parametrization of γ_1 , $\tilde{s} \mapsto s = R^{-1}(\tilde{s})$.

We need to show that $\gamma'_1(\tilde{s}) = (c - \tilde{s})\tilde{\gamma}''_1(\tilde{s})$ never vanishes on the relevant interval. Since $\tilde{s} \neq c$ for all the values we consider, it is enough to show that $\tilde{\gamma}''_1(\tilde{s})$, which is well-defined since $k-1 \geq 2$, never vanishes.

But Lemma 2 implies that $\tilde{\gamma}'_1(\tilde{s}) = \nu(R^{-1}(\tilde{s}))$. The derivative $\frac{d\nu}{ds}$ never vanishes by the hypothesis that the curvature of the initial γ is nonzero, and R^{-1} is the inverse of a $\mathcal{C}^1$ function, so $|R^{-1}(\tilde{s}_1) - R^{-1}(\tilde{s}_2)| \geq C|s_1 - s_2|$. If we had $(\nu \circ R^{-1})'(\tilde{s}_1) = 0$, then we would have

$$|\nu(R^{-1}(\tilde{s}_2)) - \nu(R^{-1}(\tilde{s}_1))| = o(|s_1 - s_2|) = o(|R^{-1}(\tilde{s}_1) - R^{-1}(\tilde{s}_2)|),$$

so $\nu'(R^{-1}(\tilde{s}_1)) = 0$, a contradiction.

Finally $\nu(s) = \tilde{\gamma}'_1(R(s))$ must be of class $\mathcal{C}^{k-2,\alpha}$. Since $\gamma'(s)$ is deduced from $\nu(s)$ by a rotation of angle $-\frac{\pi}{2}$, γ' is of class $\mathcal{C}^{k-2,\alpha}$. Now the Frenet-Serret formulae give $\nu'(s) = -\frac{1}{R(s)}\gamma'(s)$, so ν' is of class $\mathcal{C}^{k-2,\alpha}$ as a product of such. This implies that ν is of class $\mathcal{C}^{k-1,\alpha}$, so γ' , deduced from it by a rotation of angle $-\pi/2$, has the same smoothness, and γ itself is of class $\mathcal{C}^{k,\alpha}$.

The fact that the derivative of R does not vanish is equivalent to $\kappa' \neq 0$, where $\kappa = 1/R$ is the curvature of γ . The curvature of $\tilde{\gamma}$ is given by $-\kappa^3/\kappa'$; since we have assumed $\kappa(s) \neq 0$ for all s , and $\tilde{\gamma}$ is of class at least $\mathcal{C}^2$, the curvature of $\tilde{\gamma}$ is always bounded and κ' , and thus R' , do not vanish.

The fact that $R'(s) \neq 0$ will also follow from the next section, which states more precise results about the behavior of the evolute in a neighborhood of a point s_1 where γ is smooth and $R'(s_1) = 0$. $\square$

3. LOCAL BEHAVIOR

In this section, we will investigate the situations where $R'(s) \neq 0$ no longer holds. We need to quantify the vanishing of the derivative of the (radius of) curvature.

Definition 3. Let I be an open interval, $f : I \rightarrow \mathbb{R}$ be a continuous function, and $s_1 \in I$. We set

$$m_f(s_1)^- := \sup\{\mu \geq 0 : \limsup_{s \rightarrow s_1} \frac{|f(s) - f(s_1)|}{|s - s_1|^\mu} < \infty\} \in [0; \infty],$$

$$m_f(s_1)^+ := \inf\{\mu \geq 0 : \liminf_{s \rightarrow s_1} \frac{|f(s) - f(s_1)|}{|s - s_1|^\mu} > 0\} \in [0; \infty].$$

Note that $m_f(s_1)^+ \geq m_f(s_1)^-$. When f is of class $\mathcal{C}^k$, with $k \geq m_f(s_1)$, and if there exists $l \in \{1, \dots, k\}$ such that $f^{(l)}(s_1) \neq 0$,

$$\begin{aligned} m_f(s_1) &= m_f(s_1)^+ = m_f(s_1)^- \\ &= \max\{m \geq 1 : f^{(l)}(s_1) = 0, 1 \leq l \leq m\} + 1 = \min\{m \geq 1 : f^{(m)}(s_1) \neq 0\}. \end{aligned}$$

Notice that when this happens and when f is strictly monotone on a neighborhood of s_1 , $m_f(s_1)$ must be an odd integer.

With the notations from the introduction, notably a curve γ parametrized by arclength s , near a point s_1 , the *local frame* for the evolute $\tilde{\gamma}$ will be given by the point $\tilde{\gamma}(s_1)$ (which we will take to be the origin), and the direct orthonormal basis $(\nu(s_1), -\gamma'(s_1))$. By a point (x, y) we mean $\tilde{\gamma}(s_1) + x\nu(s_1) - y\gamma'(s_1)$.

Proposition 4. Let $\gamma : (a; b) \rightarrow \mathbb{R}^2$ be a $\mathcal{C}^2$ -regular curve, denote its evolute by $\tilde{\gamma}$. Let $s_1 \in (a; b)$.

- (i) Suppose that the signed radius of curvature R of γ is strictly increasing on $(a; b)$ and let $\omega(x) := R(s_1 + x) - R(s_1)$. Then $\tilde{\gamma}$ can be represented in the local frame by $y = g(x)$, where $g(x) = O(x\omega^{-1}(x))$.
- (ii) If the evolute parametrized by its arclength, $\tilde{\gamma}_1$, is $\mathcal{C}^{1,\alpha}$ -regular near s_1 for some $\alpha > 0$, then $m_R(s_1)^- \leq 1/\alpha$.
If the derivative of R vanishes to infinite order at s_1 , the curve γ is of class $\mathcal{C}^{1,0}$, and no better in the $\mathcal{C}^{1,\alpha}$ scale.
- (iii) In particular, if γ is $\mathcal{C}^k$ -regular, if R is strictly increasing on $(a; b)$ and $1 < m := m_R(s_1) \leq k$, $\tilde{\gamma}$ is $\mathcal{C}^{1,1/m}$ -regular near s_1 with $m \geq 3$, and cannot be any smoother.
- (iv) If γ is $\mathcal{C}^k$ -regular and $m := m_R(s_1)$ is a nonzero even integer, then R is not monotone in any neighborhood of s_1 and $\tilde{\gamma}$ admits a cusp: it is represented, changing ν into $-\nu$ if needed, by the union of two graphs of the form $y = g_1(x)$ and $y = -g_2(x)$, $x \geq 0$, with $g_1(x), g_2(x) \geq 0$ and $g_1(x), g_2(x) \asymp x^{1+1/m}$.

Notice that in item (i) we could have used any strictly increasing ω such that for s in a neighborhood of s_1 , $|R(s) - R(s_1)| \geq \omega(|s - s_1|)$. This can apply to cases where R is not monotone.

Proof. (i).

From (3) and the subsequent considerations we see that

$$\begin{aligned} (5) \quad \langle \tilde{\gamma}(s_2) - \tilde{\gamma}(s_1), \nu(s_1) \rangle &= (R(s_2) - R(s_1)) + \int_{s_1}^{s_2} \frac{R(s) - R(s_2)}{R(s)} \langle \gamma'(s) - \gamma'(s_1), \nu(s_1) \rangle ds \\ &= (R(s_2) - R(s_1))(1 + O(|s_2 - s_1|^2)), \\ \langle \tilde{\gamma}(s_2) - \tilde{\gamma}(s_1), \gamma'(s_1) \rangle &= \int_{s_1}^{s_2} \frac{R(s) - R(s_2)}{R(s)} \langle \gamma'(s), \gamma'(s_1) \rangle ds \\ &= O(|R(s_2) - R(s_1)||s_2 - s_1|). \end{aligned}$$

This implies in particular that the projection from the image of $\tilde{\gamma}$ to the x -axis is one-to-one, therefore we can represent that image curve as a graph $y = g(x)$; and since $x \asymp$

$R(s_2) - R(s_1)$, $R(s_2) - R(s_1) \geq \omega(s_2 - s_1)$ for $s_2 > s_1$, $R(s_2) - R(s_1) \leq -\omega(s_2 - s_1)$ for $s_2 < s_1$, and $|y| \preceq |R(s_2) - R(s_1)||s_2 - s_1|$, we have $|y| \preceq x\omega^{-1}(x)$.

(ii).

Since γ , parametrized by arclength, is (at least) $\mathcal{C}^2$ -regular, we can write $s = \gamma^{-1}(\gamma(s))$, where γ^{-1} is of class $\mathcal{C}^2$ and only defined locally on the embedded manifold $\gamma(I)$, I a small enough interval. However γ is also given as an involute of $\tilde{\gamma}$ by $\gamma_1(\tilde{s}) = \tilde{\gamma}_1(\tilde{s}) - \tilde{\gamma}'_1(\tilde{s})(s - c)$, where $\tilde{s}$ is arclength on the evolute $\tilde{\gamma}$. By the regularity hypothesis, γ_1 will be of class $\mathcal{C}^{0,\alpha}$ and $s = \gamma^{-1}(\gamma_1(\tilde{s}))$, so there is a constant $C > 0$ such that

$$|s_2 - s_1| \leq C|\tilde{s}_2 - \tilde{s}_1|^\alpha = C|R(s_2) - R(s_1)|^\alpha;$$

if $m_R(s_1)^- < \infty$, for any $\varepsilon > 0$ we have C_ε such that $|R(s_2) - R(s_1)| \leq C_\varepsilon|s_2 - s_1|^{m_R(s_1)^- - \varepsilon}$, so that $|s_2 - s_1| \leq C'_\varepsilon|s_2 - s_1|^{\alpha(m_R(s_1)^- - \varepsilon)}$, so $\alpha m_R(s_1)^- \leq 1$.

If $m_R(s_1)^- = \infty$, $|R(s_2) - R(s_1)| \leq C|s_2 - s_1|^{1+1/\alpha}$, which leads to a contradiction.

For the second statement, Theorem 1 implies that R is of class $\mathcal{C}^1$. If R was of class $\mathcal{C}^{1,\alpha}$, $\alpha > 0$, we would have $m_R(s_1)^- \leq 1/\alpha$ which contradicts the hypothesis of vanishing of the derivative to infinite order.

(iii).

The hypothesis implies that R' admits an isolated zero at s_1 , so on a punctured neighborhood of s_1 , $\tilde{\gamma}$ is of class at least $\mathcal{C}^2$ since $k \geq 3$. We have $\omega(x) \asymp x^m$ and deduce from (i) that $\tilde{\gamma}$ can be represented in the local frame by $y = g(x)$ et $g(x) = O(|x|^{1+1/m})$, in particular $\tilde{\gamma}$ is differentiable at s_1 with $\tilde{\gamma}'_1(\tilde{s}_1) = \nu(s_1)$.

More generally, for s, s' in a neighborhood of s_1 , $\tilde{\gamma}'(R(s)) = \nu(s)$ and $|\nu(s') - \nu(s_1)| \preceq |s' - s|$; we claim that

$$|s' - s| \preceq |R(s') - R(s)|^{1/m}.$$

To see this, assume without loss of generality that $R^{(m)}(t) \geq c > 0$ for t in an open interval I around s_1 , and $s, s' \in I$, $s_1 \leq s < s'$. An easy induction argument then shows that for $0 \leq j \leq m-1$, $R^{(m-j)}(s) \geq \frac{c}{j!}(s - s_1)^j \geq 0$. By the Taylor-Lagrange formula with integral remainder,

$$\begin{aligned} R(s') - R(s) &= \sum_{j=1}^{m-1} \frac{(s' - s)^j}{j!} R^{(j)}(s) + \int_s^{s'} \frac{(s' - t)^{m-1}}{(m-1)!} R^{(m)}(t) dt \\ &\geq c \int_s^{s'} \frac{(s' - t)^{m-1}}{(m-1)!} dt = c \frac{(s' - s)^m}{m!}. \end{aligned}$$

This proves the claim.

The regularity cannot be improved as a consequence of (ii).

(iv).

Now R has a strict local extremum. Suppose that it is a minimum, and that in a neighborhood of s_1 we have $R(s) - R(s_1) = c(s - s_1)^m + o(|s - s_1|^m)$, with $c > 0$. Then in the local frame, $0 \leq x = R(s) - R(s_1) \asymp (s - s_1)^m$, and for $s \geq s_1$, $y \asymp (s - s_1)^{m+1}$. For $s \leq s_1$, $-y \asymp -(s - s_1)^{m+1}$, and the claim easily follows. In the case of a local maximum, we change the sign of ν to obtain the same type of cusp in a modified local frame. $\square$

End of Proof of Theorem 1 (ii), converse part.

We already know by the proof of Theorem 1 (i) that R must be strictly monotone. If R' vanished at s_1 , we would have to have $m_R(s_1) \geq 3$, so $\tilde{\gamma}$ is at best of class $\mathcal{C}^{1,1/3}$ by Proposition 4(ii) and (iii), and cannot be of class $\mathcal{C}^2$. $\square$

4. MULTIPLE POINTS

Our definition of a regular curve restricts attention to a small enough interval in order to avoid problems with double points. Can those actually occur?

One can see that an evolute of a regular, one-to-one curve may exhibit double points.

Example 5. *There exists a $\mathcal{C}^\infty$ regular curve γ such that its evolute is a limaçon $\tilde{\gamma}$ given, for instance, by the polar representation $r(\theta) = 1 + 2 \cos \theta$, $-\frac{3\pi}{4} < \theta < \frac{3\pi}{4}$.*

The curves γ and $\tilde{\gamma}$ are represented in Figure 1.

Proof. We start from a parametric representation for an arc of limaçon, which has a double point at $(0,0)$ corresponding to the values $\theta = -2\pi/3, 2\pi/3$ of the parameter:

$$\tilde{\gamma}(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} (1 + 2 \cos \theta).$$

We deduce the arc length from $|\gamma'(\theta)| = 5 + 4 \cos \theta$. An involute for $\tilde{\gamma}$ is given by

$$\begin{aligned} \gamma(\theta) &= \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \left(1 + 2 \cos \theta + 2(\sin \theta) \frac{\int_{-\pi}^{\theta} \sqrt{5 + 4 \cos u} du}{\sqrt{5 + 4 \cos \theta}} \right) \\ &\quad - \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} (1 + 2 \cos \theta) \frac{\int_{-\pi}^{\theta} \sqrt{5 + 4 \cos u} du}{\sqrt{5 + 4 \cos \theta}} \\ &= \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} (1 + 2 \cos \theta) + \begin{pmatrix} (\sin \theta)(1 + 4 \cos \theta) \\ -4 \cos^2 \theta - \cos \theta + 2 \end{pmatrix} \frac{\int_{-\pi}^{\theta} \sqrt{5 + 4 \cos u} du}{\sqrt{5 + 4 \cos \theta}}. \end{aligned}$$

From the first formula, we easily see that γ is regular. Notice that by what follows, γ is also a simple curve. $\square$

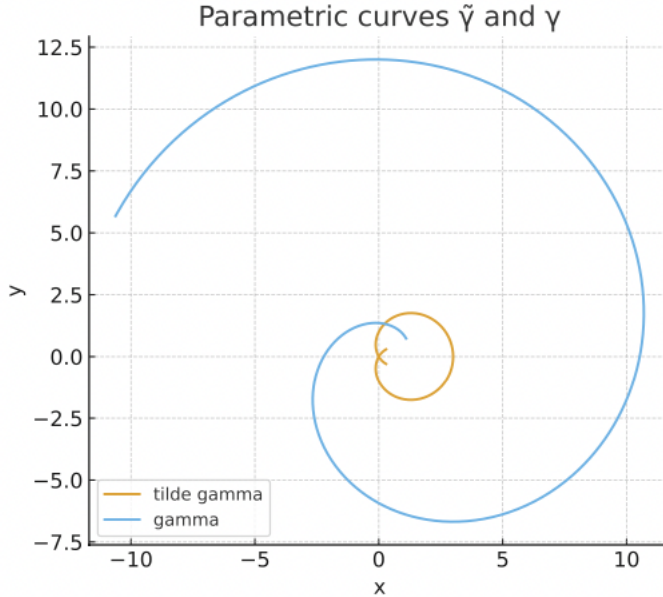


Figure 1.

Lemma 6. *The involute of a smooth curve, when it is a regular curve, has no double point.*

Recall from the proof of Theorem 1 (ii) (converse) that $\gamma_1'(\tilde{s}) = (c - \tilde{s})\tilde{\gamma}_1''(\tilde{s})$, so even if it does not start from a point on the curve $\tilde{\gamma}$, the involute could have a singularity when $\tilde{\gamma}$ undergoes an inflection.

On the other hand the same proof shows that if we start from a curve obtained as an evolute of a regular curve, the involute is regular outside of the possible points of contact with the evolute, which is not surprising since it will be a parallel to the parent curve.

Proof of Lemma 6.

By construction, the radius of curvature of an involute, being of the form $c - \tilde{s}$ must be strictly monotone. We conclude with the next result. $\square$

Lemma 7. *A smooth regular curve with strictly monotone curvature cannot have a double point.*

Proof. Let us assume that with the parametrization we have chosen, the radius of curvature is strictly increasing. We claim that near each point of the curve, $\gamma(s)$ lies inside its osculating circle at $\gamma(s_1)$ for $s < s_1$ and outside of it for $s > s_1$. Indeed the Tait-Kneser Theorem [9] tells us that osculating circles are nested within each other, so if $s_1 < s_2$, the point $\gamma(s_2)$ is on the osculating circle at $\gamma(s_2)$, and $R(s_2) > R(s_1)$, so that circle is entirely outside the osculating circle at $\gamma(s_1)$, on which lies $\gamma(s_1)$. This way $\gamma(s_2)$ can never become equal to a previously taken value. $\square$

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