OPTIMAL BOUNDS FOR THE KOBAYASHI DISTANCE NEAR $\mathcal{C}^2$-SMOOTH BOUNDARY POINTS

NIKOLAI NIKOLOV AND PASCAL J. THOMAS

Abstract. It is shown that the optimal upper and lower bounds for the Kobayashi distance near $\mathcal{C}^{2,\alpha}$-smooth strongly pseudoconvex boundary points obtained in [4] remain true in the general strongly pseudoconvex setting. In fact, the upper bound is extended to the general $\mathcal{C}^{1,1}$-smooth case. We also give upper and lower bounds for the Kobayashi distance near non-semipositive boundary points.

1. Introduction

1.1. **Motivations.** Invariant (pseudo-)distances are a powerful tool in complex analysis and geometry, and can be related to many functional properties. In spite of their concise and elegant definition (recalled below in subsection 1.3), they are almost never explicitly computable. However, obtaining estimates about their behavior helps solving questions about the geometry of domains as metric spaces, such as Gromov hyperbolicity [1], visibility [2], Gehring-Hayman type properties [5], [6], [7], and ultimately holomorphic maps.

Many estimates are known about infinitesimal invariant *metrics* near boundary points. Our focus in this paper is on the harder job of obtaining local estimates of the Kobayashi *distance* near $\mathcal{C}^2$-smooth boundary points, in the strongly pseudoconvex case (when all eigenvalues of the Levi form are positive in a neighborhood of the point) and in the non-semipositive case (when at least one eigenvalue of the Levi form is negative in a neighborhood of the point).

Recently [4] very precise upper and lower estimates were obtained for the Kobayashi distance in terms of explicit Euclidean geometric quantities for the $\mathcal{C}^{2,\alpha}$-smooth, strongly pseudoconvex domains, using methods of dilation to reduce the question to the case of the ball. We provide such estimates (Theorem 5), which are critical both near small and large distances and involve quantities of the same type for the lower and upper bounds, in the case of merely $\mathcal{C}^2$-smooth strongly pseudoconvex boundary points. Beside the reduction in the regularity required, our proof is shorter and resorts only to elementary methods.

In addition, we provide estimates in the case of pseudoconcavity (Theorems 8 and 9), which has recently proved to be of interest in the context of the question of under which conditions visibility implies pseudoconvexity [9]. The presence of only one negative eigenvalue is enough to change the situation completely, which may sound surprising, but not when one remembers that it was so already for the infinitesimal metrics [3]. We note that the distance induced by the Sibony metric will satisfy similar estimates because of the inequalities between the respective infinitesimal metrics, see [3] and references therein.

1.2. **Definitions and notations.** We will write $x \lesssim y$, or equivalently $y \gtrsim x$, when $x \le Cy$ for some constant C, and $x \asymp y$ when $x \lesssim y$ and $x \gtrsim y$. We write $\langle Z, W\rangle := \sum_{j=1}^n Z_j \overline{W}_j$ for the usual Hermitian inner product in $\mathbb{C}^n$, and $|Z|^2 := \langle Z, Z\rangle$.

The estimates that can be obtained depend on the regularity of the boundary of D. Recall that we say that ∂D is *Dini-smooth* if its unit normal is a Dini-continuous function

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of the point. Thus for any $\alpha > 0$, if ∂D is $\mathcal{C}^{1,\alpha}$-smooth, it is Dini-smooth, and if the latter property is verified, ∂D is $\mathcal{C}^1$-smooth.

Let $p \in \partial D$ be a point at which ∂D is differentiable, then n_p stands for the inner unit normal vector to ∂D.

Let $z \in D$. Then $\delta_D(z) = \operatorname{dist}(z, \partial D) = \inf_{q\in\partial D} |z - q|$. When there is a unique point $q \in \partial D$ such that $|z - q| = \delta_D(z)$, we write $q = \pi(z)$. In that case, δ_D is differentiable at z. This happens in particular when ∂D is $\mathcal{C}^{1,1}$-smooth and $\delta_D(z)$ is small enough.

At various places, we will need a notion of directional distance to the boundary:

$$\delta_D(z; X) = \sup\{\rho > 0 : z + \lambda X \in D \text{ if } |\lambda| < \rho\}. \tag{1}$$

We often need to denote the component of a vector along the complex normal to ∂D at the closest point $\pi(z)$ to z.

Definition 1. *Let D be a domain with differentiable boundary, $X \in \mathbb{C}^n$, and $z \in D$ such that $\pi(z)$ is well-defined. We set $X_z = 2\langle X, \bar\partial\delta_D(z)\rangle = \langle X, n_{\pi(z)}\rangle$.*

Our results will be stated in terms of the following auxiliary quantities.

$$B_D(z, w) = |(z - w)_z| + |z - w|^2 + |z - w|\sqrt{\delta_D(z)}, \quad A_D(z, w) = \frac{B_D(z, w)}{\sqrt{\delta_D(z)\delta_D(w)}}. \tag{2}$$

By [4, Remark 7], $A_D(z, w) \asymp A_D(w, z)$ (see also (11) below).

1.3. **Invariant quantities.** Let D be a domain in $\mathbb{C}^n$, $z, w \in D$ and $X \in \mathbb{C}^n$. Denote by $\mathbb{D}$ the unit disc. The Kobayashi-Royden (pseudo)metric κ_D and the Lempert function are defined as follows:

$$\kappa_D(z; X) = \inf\{|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D}, D), \varphi(0) = z, \alpha\varphi'(0) = X\},$$
$$l_D(z, w) = \inf\{\tanh^{-1}|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \varphi(\alpha) = w\}.$$

The Kobayashi (pseudo)distance k_D is the largest pseudodistance not exceeding l_D. It turns out that k_D is the integrated form of κ_D, that is,

$$k_D(z, w) = \inf_\gamma \int_0^1 \kappa_D(\gamma(t); \gamma'(t))dt,$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, 1] \to D$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

1.4. **Previous results.** The main problem about the boundary behavior of $k_D(z, w)$ and $l_D(z, w)$ is the case when z and w approach the same point.

In [8, Theorem 7], the following upper bound for k_D near Dini-smooth boundary points of D is proved:

$$k_D(z, w) \le \log\left(1 + \frac{2|z - w|}{\sqrt{\delta_D(z)\delta_D(w)}}\right).$$

Much more is known in the strongly pseudoconvex case.

Denote by cc_D the Carnot-Carathéodory distance on ∂D near a strongly pseudoconvex point (the sub-Riemannian structure where the horizontal curves are taken to be the complex tangential ones, see e.g. [1, p. 505]).

From [4, (4) and (6)], we see that

$$(1 + A_D(z, w))\delta_D(z)^{1/2}\delta_D(w)^{1/2} \asymp cc_D(\pi(z), \pi(w))^2 + \max(\delta_D(z), \delta_D(w)), \tag{3}$$

So, assuming $|X_z| = |X|$ if $\delta_D(z) \gtrsim 1$, the well-known Balogh-Bonk estimate [1, Corollary 1.3] can be also read as follows.

Proposition 2. *For any strongly pseudoconvex domain D there exists $C > 0$ such that*

$$\log(1 + A_D) - C \le k_D \le \log(1 + A_D) + C. \tag{4}$$

Using a different approach based on complex geodesics, [4, Corollary 8] refines (4) (when $A_D \lesssim 1$) in the $\mathcal{C}^{2,\alpha}$-smooth case.

Proposition 3. *If $p \in \partial D$ is strongly pseudoconvex and $\mathcal{C}^{2,\alpha}$-smooth, then there exist $0 < c < C$ such that near p one has that*

$$\log(1 + cA_D) \le k_D \le \tag{5}$$

$$l_D \le \log(1 + CA_D). \tag{6}$$

2. Results

There is a partial converse to Proposition 3.

Proposition 4. *If* (5) *holds near a $\mathcal{C}^2$-smooth point $p \in \partial D$, then p is strongly pseudoconvex.*

This is proved in Section 3.

Our goal is to extend Propositions 2 and 3.

Theorem 5. (5) *and* (6) *hold near any strongly pseudoconvex point.*

The proof of (5) is given in Section 3.

In fact, (6) is an easy consequence of the following general statement. Set

$$\chi_{D,p} = \limsup_{\partial D \ni p', p'' \to p} \frac{|n_{p'} - n_{p''}|}{|p' - p''|} < \infty,$$

$$A_{D,p}(z,w) = \frac{|(z-w)_z| + \chi_{D,p}|z-w|^2 + |z-w|\sqrt{\chi_{D,p}\delta_D(z)}}{\sqrt{\delta_D(z)\delta_D(w)}}.$$

Theorem 6. *There exists $C_0 > 0$ such that for any domain D (in any $\mathbb{C}^n$) and any $\mathcal{C}^{1,1}$-smooth point $p \in \partial D$ with $\chi_{D,p} > 0$ one may find a neighborhood U of p such that*

$$k_D(z,w) \le \log(1 + C_0 A_{D,p}(z,w)), \quad z, w \in D \cap U. \tag{7}$$

It follows immediately from the proof of Theorem 6, given in Section 4 below, that if $\chi_{D,p} = 0$, then the same holds replacing $\chi_{D,p}$ in $A_{D,p}$ by any $\varepsilon > 0$.

Recall now that the Carathéodory (pseudo)distance is defined by

$$c_D(z,w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D,\mathbb{D}), f(z) = 0\}.$$

One can deduce from Theorem 5 its global version corresponding to [4, Theorem 1]. We skip the proof because the arguments are the same as in [4].

Proposition 7. *For any strongly pseudoconvex domain D there exist $0 < c < C$ such that*

$$\log(1 + cA_D) \le c_D \le k_D \le l_D \le \log(1 + CA_D).$$

Finally, we will consider the opposite case to strong pseudoconvexity. Recall that $s : \mathbb{C}^n \longrightarrow \mathbb{R}$ is a *defining function* for D if $D = \{z \in \mathbb{C}^n : s(z) < 0\}$. We call a $\mathcal{C}^2$-smooth point $p \in \partial D$ *non-semipositive* if, for s a defining function of D (it does not depend which one), $\partial\overline{\partial}s(X,\overline{X})(p) < 0$ for some vector $X \in T_p^{\mathbb{C}}(\partial D)$.

The *signed distance* to ∂D is the special defining function given by $r_D = -\delta_D$ on D and $r_D = \delta_D$ on $\mathbb{C}^n \setminus D$.

Set

$$H_D(z,w) = \frac{|(z-w)_z|}{\sqrt{|(z-w)_z| + |z-w| + \sqrt{\delta_D(z)}}} + |z-w|. \tag{8}$$

Note that $H_D(z,w) \asymp \frac{B_D(z,w)}{\sqrt{|(z-w)_z|+|z-w|+\sqrt{\delta_D(z)}}}$ with B_D as in (2). First notice that $H_D(z,w) \asymp H_D(w,z)$.

Indeed, it already follows from (11) that $B_D(z,w) \asymp B_D(w,z)$. Now consider the denominator $\sqrt{\delta_D(z)} + \sqrt{|(z-w)_z|} + |z-w|$.

From (10), $\sqrt{|(z-w)_z|} \le \sqrt{|(z-w)_w|} + |z-w|$, and from (14),

$$\delta_D(z) \le \delta_D(w) + |(z-w)_z| + |z-w|^2,$$

so that

$$\sqrt{\delta_D(z)} + \sqrt{|(z-w)_z|} + |z-w| \le 3\left(\sqrt{\delta_D(w)} + \sqrt{|(z-w)_w|} + |z-w|\right),$$

then switching the variables z, w, we get the result.

The quantity provides a holds regardless of pseudoconvexity hypotheses (and is of course much smaller than the one in (5), so far from optimal in the strongly pseudoconvex case).

Theorem 8. *Let p be a $\mathcal{C}^{1,1}$-smooth point in ∂D, then for z, w close enough to p, $k_D(z,w) \gtrsim H_D(z,w)$.*

In particular, $k_D(z,w) \gtrsim |z-w| + |\sqrt{\delta_D(z)} - \sqrt{\delta_D(w)}|$ (see Proposition 15).

Theorem 9. *Let p be a non-semipositive point in ∂D, then for z, w close enough to p, $k_D(z,w) \lesssim H_D(z,w)$.*

In particular, near any non-semipositive point, $k_D(z,w) \asymp H_D(z,w)$ and $k_D(z,w) \lesssim \sqrt{|(z-w)_z|} + |z-w|$.

Theorems 8 and 9 are proved in Section 5.

3. Proof of Theorem 5

Henceforth, we will always assume that $z, w \in D$ are close enough to $p \in \partial D$.

We start with the proof of the easier converse direction.

Proof of Proposition 4. It follows by [10, Proposition 2] that

$$(5) \Rightarrow l_D(z,w) \ge \log(1 + cA_D(z,w)) \Rightarrow \kappa_D(z;X) \ge c\frac{|X|}{\sqrt{\delta_D(z)}}.$$

We will show that the last implies that p is strongly pseudoconvex.

There exist a neighborhood U of p and a biholomorphism $\Phi : U \to \Phi(U)$ such that, if we write $G = \Phi(D \cap U)$, and r_G for the signed distance function to ∂G, the terms coming from $\partial^2 r_G$ and $\bar\partial^2 r_G$ vanish in the real Hessian $H(r_G)$ of r_G at $\Phi(p)$, so that $H(r_G)$ is positive definite. Then one may find $c' > 0$ for which

$$\kappa_G(z;X) \ge c'\frac{|X|}{\sqrt{\delta_G(z)}}, \quad z \text{ near } \Phi(p).$$

So we can reduce to the case where our domain is G, which we denote again by D.

Assume that p is not strongly pseudoconvex, that is, $\partial\bar\partial r_D(X, \overline{X})(p) \le 0$ for some unit vector $X \in T_p^{\mathbb{C}}(\partial D)$. Denote by N_p the inner normal to ∂D at p. Then

$$\lim_{N_p \ni z \to p} \frac{\sqrt{\delta_D(z)}}{\delta_D(z;X)} = 0,$$

where $\delta_D(z;X)$ is as in (1).

On the other hand,

$$\kappa_D(z;X) \le \frac{1}{\delta_D(z;X)}$$

which is a contradiction. □

We need to study the behavior of the quantity A_D defined in (2) under biholomorphisms and interversion of variables. To do so, it is convenient to introduce the following slightly more general quantity: for any $\mathcal{C}^{1,1}$-smooth defining function s of D near $p \in \partial D$, define A_D^s and B_D^s as in (2), replacing δ_D by $-s$ throughout.

Proposition 10. $A_D^s \asymp A_D$.

Proof. Note that $s = h\delta_D$, where $h < 0$ is Lipschitz near p. Then $\bar\partial s = h\bar\partial\delta_D + O(\delta_D)$ which easily implies that $A_D^s \asymp A_D$. □

Proposition 11. *Let $D, G \Subset \mathbb{C}^n$ and $f : \overline{D} \to \overline{G}$ be a biholomorphism. If ∂D is $\mathcal{C}^{1,1}$-smooth, then for $z, w \in D$,*

$$A_D(z,w) \asymp A_G(f(z), f(w)).$$

Proof. Note that $s := -\delta_D \circ f^{-1}$ is a defining function of G and

$$\begin{aligned}(9)\quad \langle f(w) - f(z), \bar\partial s(f(z))\rangle &= \langle Jf(z)(z-w) + O(|z-w|^2), ((Jf(z))^{-1})^* \bar\partial\delta_D(z)\rangle \\ &= \langle z - w, \bar\partial\delta_D(z)\rangle + O(|z-w|^2)\end{aligned}$$

(the vectors are column vectors).

Then (9) and $|f(z) - f(w)| \asymp |z - w|$ imply that $A_D(z,w) \asymp A_G^s(f(z), f(w))$. By Proposition 10, we are done. □

Proof of Theorem 5. As above, there exist a neighborhood U of p and a biholomorphism $\Phi : U \to \Phi(U)$ such that $G := \Phi(D \cap U)$ is a strongly convex domain.

By [4, Lemma 20] (see also Proposition 11),

$$A_D(z,w) = A_{D\cap U}(z,w) \asymp A_G(\Phi(z), \Phi(w)).$$

On the other hand, by [11, Theorem 1.1],

$$k_D \le k_{D\cap U} < k_D + C, \quad k_{D\cap U} \asymp k_D.$$

So we may replace D by G, which we again relabel D.

Then $c_D = k_D = l_D$ by Lempert's theorem and so (6) follows from Theorem 6.

Now we will prove (5). The case $A_D(z,w) \gtrsim 1$ is covered by the LHS inequality in (4).

Further, since $A_D(z,w) \asymp A_D(w,z)$, we may also assume that $\delta_D(w) \ge \delta_D(z)$. Then, by [13, p. 633],

$$k_D(z,w) \ge \frac{1}{2}\log\left(1 + \frac{1}{\delta_D(z; z-w)}\right).$$

Since D can be placed between two balls having $\pi(z) \in \partial D$ as a boundary point and radii independent of $\pi(z)$, it follows that

$$\frac{1}{\delta_D(z; z-w)} \asymp \hat{A}_D(z,w) := \frac{|(z-w)_z|}{\delta_D(z)} + \frac{|z-w|}{\sqrt{\delta_D(z)}}.$$

Note that [4, Proposition 22] implies that

$$\hat{A}_D(z,w) \asymp \frac{B_D(z,w)}{\delta_D(z)} \ge A_D(z,w).$$

So $k_D(z,w) \gtrsim A_D(z,w)$ if $A_D(z,w) \lesssim 1$. □

4. Proof of Theorem 6

Applying a homothety will modify $\chi_{D,p}$ and only change the constants in (7), so we may set $\chi_{D,p} = \chi_0 \in (0,1)$.

Let $E_{q,r}$ denotes the ball with center $q_* := q + rn_q$ and radius r. With our choice of $\chi_{D,p} < 1$, there exists a neighborhood V of p such that for $q \in \partial D$ near p, $E_{q,1} \cap V \subset D$.

It is enough to prove (7) with A_D instead of $A_{D,p}$.

Set $X = z - w$, so that $B_D(z,w) = |X_z| + |X|^2 + |X|\sqrt{\delta_D(z)}$.

Since $\delta_D(w) \le \delta_D(z) + |z-w|$ and

$$|X_w| - |X_z| \le |X_w - X_z| = \left|\langle X, n_{\pi(z)} - n_{\pi(w)}\rangle\right| \le (\chi_0 + o(1))|X|^2 \le |X|^2, \tag{10}$$

because $|\pi(z) - \pi(w)| \le (\chi_0 + o(1))|z-w|$. It follows that

$$A_D(w,z) \le 2A_D(z,w). \tag{11}$$

So we may assume for the remainder of the proof that $\delta_D(w) \le \delta_D(z)$ (note that this is the opposite convention to the previous proof). It is enough to show the following.

Claim 12. *(a) $k_D(z,w) \le 4A_D(z,w)$ if $2B_D(z,w) \le \sqrt{\delta_D(z)\delta_D(w)}$;*
(b) $k_D(z,w) < \log A_D(z,w) + 10$ if $2B_D(z,w) > \sqrt{\delta_D(z)\delta_D(w)}$.

To prove (a), let γ be the line segment from z to w, explicitly $\gamma(t) = z - tX$, $t \in [0,1]$, and recall that now $\pi(z)_* = \pi(z) + n_{\pi(z)}$. Then

$$\begin{aligned} 1 - |\gamma(t) - \pi(z)_*|^2 &= 1 - \left|\pi(z) + \delta_D(z)n_{\pi(z)} - tX - (\pi(z) + n_{\pi(z)})\right|^2 \\ &= 1 - \left|(1-\delta_D(z))n_{\pi(z)} + tX\right|^2 \ge 2\delta_D(z) - \delta_D^2(z) - |X|^2 - 2(1-\delta_D(z))|\mathrm{Re}X_z| \\ &\ge 3\delta_D(z)/2 - 2B_D(z,w) \ge \delta_D(z)/2; \end{aligned} \tag{12}$$

hence $\gamma(t) \in D$ and $\delta_D(\gamma(t)) \ge \delta_{E_{\pi(\gamma(t)),1}}(\gamma(t)) > \delta_D(z)/4$.

Since γ is a competitor for the Kobayashi distance, $k_D(z,w) \le \int_0^1 \kappa_D(\gamma(t);X)dt$ and there exists $u = \gamma(t_0)$ such that

$$k_D(z,w) \le \kappa_D(u;X).$$

Since $E_{\pi(u),1} \cap V \subset D$, then

$$\kappa_D(u;X) \le \frac{1}{\delta_D(u;X)} \le \frac{|X_u|}{\delta_D(u)} + \frac{|X|}{\sqrt{\delta_D(u)}} \le \frac{4|X_u|}{\delta_D(z)} + \frac{2|X|}{\sqrt{\delta_D(z)}}. \tag{13}$$

Here we have used (12) and the fact that $\pi(u) \in (\partial E_{\pi(u),1}) \cap (\partial D)$, and that $n_{\pi(u)}$ is the common normal unit vector to $\partial E_{\pi(u),1}$ and ∂D at that point. Using that

$$|X_u| - |X_z| \le |X| \cdot |u-z| \le |X|^2,$$

we conclude that

$$k_D(z,w) \le \frac{4B_D(z,w)}{\delta_D(z)} \le 4A_D(z,w).$$

To prove (b), we need a technical result about the behaviour of the distance function. Write $x \cdot y := \sum_{j=1}^m x_j y_j$ for the Euclidean scalar product in $\mathbb{R}^m$. When $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ in the usual way, this means that $Z \cdot W = \mathrm{Re}\langle Z, W\rangle$.

Proposition 13. [12, Proposition 8] *Let p be a $\mathcal{C}^{1,1}$-smooth boundary point of a domain D in $\mathbb{R}^m$. Then the signed distance r_D to ∂D is a $\mathcal{C}^{1,1}$-smooth function near p. Moreover, for x, y near p one has that*

$$|r_D(x) - r_D(y) - \nabla r_D(x)\cdot(x-y)| \le (\chi_{D,p}/2 + o(1))(|x-y|^2 - (r_D(x) - r_D(y))^2).$$

Since we have set $\chi_0 < 1$, this implies

$$\delta_D(z) - \delta_D(w) \le \mathrm{Re}\,X_z + (\chi_0/2 + o(1))|X|^2 \le B_D(z,w). \tag{14}$$

Then the assumption on $A_D(z,w)$ implies $4B_D(z,w)^2 > \delta_D(z)\delta_D(w)$, which because of (14) is $\ge \delta_D(z)(\delta_D(z) - B_D(z,w))$, from which one deduces $3B_D(z,w) > \delta_D(z)$.

Let

$$z' = \pi(z) + 3B_D(z,w)n_{\pi(z)} \text{ and } w' = \pi(w) + 3B_D(z,w)n_{\pi(w)}, \text{ then} \tag{15}$$

$$k_D(z,w) \le k_D(z,z') + k_D(w,w') + k_D(z',w').$$

Choose $r > 0$ such that $E_{q,r} \subset D$ for any $q \in \partial D$ near p. Then an explicit calculation of the invariant distances in the unit ball leads to

$$k_D(z,z') \le k_{E_{\pi(z)},r}(z,z') \le \log \frac{\sqrt{\delta_D(z')}}{\sqrt{\delta_D(z)}} + o(1) = \log \frac{\sqrt{3B_D(z,w)}}{\sqrt{\delta_D(z)}} + o(1),$$

$$k_D(w,w') \le k_{E_{\pi(w)},r}(w,w') \le \log \frac{\sqrt{\delta_D(w')}}{\sqrt{\delta_D(w)}} + o(1) = \log \frac{\sqrt{3B_D(z,w)}}{\sqrt{\delta_D(w)}} + o(1),$$

and hence

$$k_D(z,z') + k_D(w,w') \le \log(3A_D(z,w)) + o(1).$$

To estimate $k_D(z',w')$, set $X' = z' - w'$. Writing $u' = z' - tX'$, we have as above that

$$1 - |u' - \pi(z)_*|^2 = 1 - \left|(1-3B_D)n_{\pi(z)} + tX'\right|^2 \ge (6+o(1))B_D(z,w) - |X'|^2 - 2|X'_z|.$$

By using (14) and $|n_{\pi(z)} - n_{\pi(w)}| = O(|z-w|) = o(1)$, we find

(16)
$$\begin{aligned}|X'-X| = |(z-z')-(w-w')| &= \left|(3B_D(z,w) - \delta_D(z))(n_{\pi(z)} - n_{\pi(w)}) + (\delta_D(z) - \delta_D(w))n_{\pi(w)}\right| \\ &\le (3B_D(z,w) - \delta_D(z))|n_{\pi(z)} - n_{\pi(w)}| + \delta_D(z) - \delta_D(w) \le (1+o(1))B_D(z,w).\end{aligned}$$

Since $|X|^2 \le B_D(z,w)$, it follows from the triangle inequality that

$$|X'|^2 \le (1 + O(B_D(z,w)^{1/2}))B_D(z,w) = (1+o(1))B_D(z,w). \tag{17}$$

Also,

$$|X'_z| \le |X_z| + |X - X'| \le (2+o(1))B_D(z,w) \tag{18}$$

and so

$$1 - |u' - \pi(z)_*|^2 \ge (6+o(1))B_D(z,w) - |X'|^2 - 2|X'_z| \ge (1+o(1))B_D(z,w).$$

Hence $u' \in D$ and

$$\delta_D(u') \ge \delta_{E_{\pi(u),1}}(u') \ge (1/2 + o(1))B_D(z,w). \tag{19}$$

Then as in the proof of (a), one may find $u' \in [z'; w']$ such that

$$k_D(z',w') \le \kappa_D(u';X') \le \frac{|X'_{u'}|}{\delta_D(u')} + \frac{|X'|}{\sqrt{\delta_D(u')}}.$$

Since $|X'_{u'}| - |X'_{z'}| \le |X'|^2$ and $X'_{z'} = X'_z$, it follows by (17), (18), and (19) that

$$k_D(z',w') < 6 + \sqrt{2} + o(1) < 10 - \log 3.$$

So (b) is proved. □

5. Proofs of Theorems 8 and 9

5.1. **Proof of Theorem 9.** Using a homothety (thus changing the constants if needed), we may assume that $\chi_{D,p} < 1$.

Since $H_D(z,w) \asymp H_D(w,z)$, we may assume that $\delta_D(z) \ge \delta_D(w)$.

We will show that $k_D(z,w) \lesssim H_D(w,z)$. By [3, Proposition 1], near p,

$$\hat{\kappa}_D(u;X) \lesssim \frac{|X_u|}{\sqrt{\delta_D(u)}} + |X|. \tag{20}$$

Define z' and w' as in (15).

Claim 14. *(a)* $k_D(z,w) \lesssim \frac{B_D(z,w)}{\sqrt{\delta_D(z)}}$ *if* $2B_D(z,w) \le \delta_D(z)$;

(b) $k_D(z,w) \le k_D(z,z') + k_D(w,w') + k_D(z',w') \lesssim \sqrt{B_D(z,w)}$ *if* $2B_D(z,w) > \delta_D(z)$.

Proof of Claim 14. We follow the general scheme of the proof of Claim 12, and reuse those of the arguments which only depend on the smoothness of ∂D.

To prove (a), we define X, γ, u and $E_{\pi(u),1}$ as before and follow the proof until before (13). Instead of (13), we use (20), and since $\delta_D(u) \geq \frac{1}{4}\delta_D(z)$ and $|X_u| - |X_z| \leq |X|^2$,

$$\hat{\kappa}_D(u; X) \leq 2\frac{|X_z| + |X|^2}{\sqrt{\delta_D(z)}} + |X| \leq 2\frac{B_D(z,w)}{\sqrt{\delta_D(z)}}.$$

To prove (b), we do not need to use (14) to bound $B_D(z,w)$ from below, since $2B_D(z,w) > \delta_D(z)$ by assumption.

To estimate $k_D(z,z')$ and $k_D(w,w')$, simply integrate along the normal line to get as upper bound $c\int_{\delta_D(z)}^{3B_D(z,w)} t^{-1/2}dt \lesssim B_D(z,w)^{1/2}$ (the same computation for w).

Then we use $k_D(z',w') \lesssim |z'-w'|$, and by the definition of $B_D(z,w)$,

$$|z'-w'| \leq |z'-z|+|z-w|+|w-w'| \leq 3B_D(z,w)+\sqrt{B_D(z,w)}+3B_D(z,w) \lesssim \sqrt{B_D(z,w)}.$$

□

End of the proof of $k_D(z,w) \lesssim H_D(w,z)$. In case (a), the assumption implies $|(z-w)_z| + |z-w|^2 \leq \delta_D(z)/2$, so that $\sqrt{|(z-w)_z|} + |z-w| \leq \sqrt{\delta_D(z)}$, and

$$\frac{B_D(z,w)}{\sqrt{\delta_D(z)}} \leq 2H_D(z,w).$$

In case (b), the assumption implies $\sqrt{B_D(z,w)} \gtrsim \sqrt{|(z-w)_z|} + |z-w| + \sqrt{\delta_D(z)}$, so that

$$\sqrt{B_D(z,w)} \lesssim \sqrt{B_D(z,w)}\frac{\sqrt{B_D(z,w)}}{\sqrt{|(z-w)_z|} + |z-w| + \sqrt{\delta_D(z)}} \leq H_D(z,w).$$

□

5.2. **Proof of Theorem 8.** We start with a first lower bound for k_D.

Proposition 15. *If D is bounded, then $k_D \gtrsim F_D$ near a $\mathcal{C}^{1,1}$-smooth point $p \in \partial D$, where $F_D(z,w) = |z-w| + |\sqrt{\delta_D(z)} - \sqrt{\delta_D(w)}|$.*

Proof. We will use that k_G is also the integrated form of the Kobayashi-Buseman (pseudo)metric $\hat{\kappa}_G$ – the largest (pseudo)metric not exceeding κ_G with convex indicatrices.

Let $D_R := B(0,R) \setminus \overline{B}(0,1)$. By [3, Proposition 1],

$$\hat{\kappa}_{D_R}(u; X) \gtrsim \frac{|X_u|}{\sqrt{\delta_{D_R}(u)}} + |X|. \tag{21}$$

Given a $\mathcal{C}^2$ point p, by continuity of the second derivatives, there exists a neighborhood V of p and a radius $R_0 > 0$ such that for any $p' \in V \cap \partial D$, the ball $B_{p'} := B(p' - R_0 n_{p'}, R_0)$ verifies $B_{p'} \cap D = \emptyset$, $B_{p'} \cap \overline{D} = \{p'\}$. Using that D is bounded, there exist $R > 1$ and a neighborhood U of p such that for any $q \in U \cap \partial D$, after a homothety and a translation, one has that

$$D \subset D_R \text{ and } q \in \partial B(0,1). \tag{22}$$

Consider $z, w \in U$, close enough to p so that $q := \pi(w) \in U$. We may assume that $\delta_D(z) \geq \delta_D(w)$. We choose and find D_R as above. Since D_R and D must have the same tangent space at q, in our new coordinates, $q = \frac{w}{|w|}$.

Let γ be a curve from z to w in D such that $\gamma(0) = z$ and $\gamma(1) = w$. Then

$$\int_0^1 \hat{\kappa}_{D_R}(\gamma(t); \gamma'(t))dt \gtrsim \int_0^1 \left(|\gamma'(t)| + \frac{\mathrm{Re}\langle\gamma'(t), \gamma(t)\rangle}{(|\gamma(t)|^2 - 1)^{1/2}} \right) dt = \int_0^1 |\gamma'(t)|dt + \frac{1}{2}\int_{|w|^2}^{|z|^2} \frac{dx}{\sqrt{x-1}}$$
$$\geq |z-w| + \sqrt{|z|^2-1} - \sqrt{|w|^2-1} \geq |z-w| + \sqrt{\delta_D(z)(2+\delta_D(z))} - \sqrt{\delta_D(w)(2+\delta_D(w))}$$

$$\geq |z-w| + \sqrt{2\delta_D(z)} - \sqrt{2\delta_D(w)}.$$

So $k_D(z,w) \geq k_{D_R}(z,w) \gtrsim |z-w| + \sqrt{\delta_D(z)} - \sqrt{\delta_D(w)}$. □

It turns out that Proposition 15 is the "real" version of Theorem 8. Indeed, define H_D^r by replacing $(z-w)_z$ by $\mathrm{Re}(z-w)_z$ in equation (8):

$$H_D^r(z,w) = \frac{|\mathrm{Re}(z-w)_z|}{\sqrt{|\mathrm{Re}(z-w)_z| + |z-w| + \sqrt{\delta_D(z)}}} + |z-w|.$$

Proposition 16. *$F_D \asymp H_D^r \leq H_D$ near any $\mathcal{C}^{1,1}$-smooth point $p \in \partial D$.*

Proof. The inequality $H_D^r(z,w) \leq H_D(z,w)$ is equivalent to $|\mathrm{Re}(z-w)_z| \leq |(z-w)_z|$.

Further, observe that

$$F_D(z,w) < \sqrt{\delta_D(z)} + \sqrt{\delta_D(w)} + |z-w| \text{ and } \sqrt{|\mathrm{Re}(z-w)_z|} + |z-w| \leq H_D^r(z,w).$$

So, if $\sqrt{\delta_D(z)} + \sqrt{\delta_D(w)} \leq |\mathrm{Re}(z-w)_z| + |z-w|$, then $F_D(z,w) < 2H_D^r(z,w)$.

Otherwise,

$$|\sqrt{\delta_D(z)} - \sqrt{\delta_D(w)}| = \frac{|\delta_D(z)-\delta_D(w)|}{\sqrt{\delta_D(z)}+\sqrt{\delta_D(w)}} \leq \frac{|\mathrm{Re}(z-w)_z| + C^2|z-w|^2}{\sqrt{\delta_D(z)}+\sqrt{\delta_D(w)}}$$
$$\leq \frac{2|\mathrm{Re}(z-w)_z|}{\sqrt{\delta_D(z)} + \sqrt{|\mathrm{Re}(z-w)_z| + |z-w|}} + C^2|z-w|.$$

where the first inequality follows from Proposition 13.

Thus $F_D(z,w) \leq \max\{2, C^2+1\} H_D^r(z,w)$ in both cases.

Similarly, by Proposition 13, $\sqrt{\delta_D(w)} \leq \sqrt{\delta_D(z)} + \sqrt{|\mathrm{Re}(z-w)_z|} + C|z-w|$, so

$$|\sqrt{\delta_D(z)} - \sqrt{\delta_D(w)}| \geq \frac{|\mathrm{Re}(z-w)_z| - C^2|z-w|^2}{2\sqrt{\delta_D(z)} + 2\sqrt{|\mathrm{Re}(z-w)_z|} + C|z-w|} \geq C^{-1}H_D^r(z,w) - C|z-w|,$$

where $C \geq 2$. This and $F_D(z,w) \geq |z-w|$ imply that $C(C+1)F_D(z,w) \geq H_D^r(z,w)$. □

To get the full statement of Theorem 8, we reduce ourselves to the case of the set difference of two balls, using (22) and the considerations that precede it. This time we will choose coordinates such that (after translation) $|z| \geq |w|$ and $\pi(z) = \frac{z}{|z|}$, and after an additional rotation, $\frac{z}{|z|} = (1, 0, \dots, 0)$. Notice that then $H_D(z,w) = H_{D_R}(z,w)$, since n_z is the same whether it is taken with respect to D or to D_R. Since $k_D(z,w) \geq k_{D_R}(z,w)$, the estimate we seek reduces to the following.

Proposition 17. *Let $z = (1+\varepsilon_1, 0) \in D_R \subset \mathbb{C} \times \mathbb{C}^{n-1}$, let $w \in D_R$, with $|w| \leq |z|$. Then $k_{D_R}(z,w) \gtrsim H_{D_R}(z,w)$.*

Proof. Throughout this proof, we will simply write $D_R = D$.

A further rotation within the plane $\{0\} \times \mathbb{C}^{n-1}$ allows us to write $w = (1+\varepsilon_2)(e^{i\eta}\cos\beta, \sin\beta, 0, \dots, 0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-2}$, with $0 < \varepsilon_2 \leq \varepsilon_1$, and $0 \leq \beta \leq \pi/2$. We only need the estimate for nearby points, so that we may assume $\beta + (\varepsilon_1 - \varepsilon_2) + |\eta| \asymp |z-w| \ll 1$.

Then $\pi(z) = (1,0)$, $\pi(w) = (e^{i\eta}\cos\beta, \sin\beta, 0, \dots, 0)$,

$$(z-w)_z = (1+\varepsilon_1) - (1+\varepsilon_2)e^{i\eta}\cos\beta =$$
$$(\varepsilon_1 - \varepsilon_2) + 2(1+\varepsilon_2)(1 - 2\sin^2\frac{\eta}{2})\sin^2\frac{\beta}{2} + 2(1+\varepsilon_2)\sin^2\frac{\eta}{2} + i(1+\varepsilon_2)\cos\beta\sin\eta.$$

Thus $|\mathrm{Re}(z-w)_z| \asymp (\varepsilon_1-\varepsilon_2)+\eta^2+\beta^2$, $|\mathrm{Im}(z-w)_z| \asymp |\eta|$, $|(z-w)_z| \asymp (\varepsilon_1-\varepsilon_2)+|\eta|+\beta^2$.
This implies

$$H_D(z,w) \asymp \frac{(\varepsilon_1-\varepsilon_2) + |\eta| + \beta^2 + \beta\sqrt{\varepsilon_1}}{\sqrt{|\eta|} + \beta + \sqrt{\varepsilon_1}},$$

while

$$H^r_D(z,w) \asymp |\eta| + \beta + \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}.$$

When $|\operatorname{Re}(z-w)_z| \asymp |(z-w)_z|$, then $H^r_D(z,w) \asymp H_D(z,w)$, so we only need to deal with the cases where this does not happen, i.e. $(\varepsilon_1 - \varepsilon_2) + \beta^2 \le c_0|\eta|$, where $c_0 \le 10^{-2}$ is an absolute constant, the value of which will be fixed later.

With this new requirement,

$$H_D(z,w) \asymp \frac{|\eta| + \beta\sqrt{\varepsilon_1}}{\sqrt{|\eta|} + \sqrt{\varepsilon_1}}.$$

When $|\eta| \le \varepsilon_1$, this reduces to $H_D(z,w) \asymp \beta + \frac{|\eta|}{\sqrt{\varepsilon_1}}$; when $|\eta| \ge \varepsilon_1$, this reduces to $H_D(z,w) \asymp \beta + \sqrt{|\eta|}$; so altogether, there are two absolute constants $C_1 < C_2$ so that for any choice of $c_0 \le 10^{-2}$ we have

$$C_1 \left(\beta + \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}}\right)\right) \le H_D(z,w) \le C_2 \left(\beta + \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}}\right)\right). \tag{23}$$

Let γ be an absolutely continuous curve such that $\gamma(0) = z$, $\gamma(1) = w$. Suppose that γ is close enough to minimize the integral of the Kobayashi-Busemann metric. Since for any t, $k_D(z,\gamma(t)) \lesssim k_D(z,w)$, and since $H^r_D \lesssim k_D \lesssim H_D$,

$$|z - \gamma(t)| + \left|\sqrt{\delta_D(z)} - \sqrt{\delta_D(\gamma(t))}\right| \lesssim \beta + \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}}\right),$$

so that for ε, η small enough, $\delta_D(\gamma(t)) = |\gamma(t)| - 1$ and $\delta_D(\gamma(t)) \lesssim \varepsilon_1 + \beta^2 + |\eta|$.

With the above restrictions on the values of $\delta_D(\gamma(t))$, we can write

$$\gamma(t) = (1 + \rho(t))\left(e^{i\theta_1(t)} \cos\alpha(t), \zeta(t) \sin\alpha(t)\right),$$

where ρ, α, θ_1 are absolutely continuous real-valued functions, and $\zeta(t) \in \mathbb{C}^{n-1}$, $|\zeta(t)| = 1$ for all t. All functions are also a.e. differentiable, and the condition on ζ implies $\operatorname{Re}\langle\zeta(t), \zeta'(t)\rangle = 0$ for all t where it makes sense.

We must have $\rho(t) > 0$ for all t, $\rho(0) = \varepsilon_1$, $\rho(1) = \varepsilon_2$, $\theta_1(0) = 0$, $\theta_1(1) = \eta$, $\zeta(1) = (1, 0, \dots, 0)$, $\alpha(0) = 0$, $\alpha(1) = \beta$.

Write $\gamma^*(t) := \pi(\gamma(t)) = \gamma(t)/|\gamma(t)|$, so that again $\operatorname{Re}\langle\gamma^{*\prime}(t), \gamma^*(t)\rangle = 0$. We compute

$$\begin{aligned} (24)\quad \gamma'(t) &= \rho'(t)\gamma^*(t) + (1+\rho(t))\gamma^{*\prime}(t) \\ &= \rho'(t)\gamma^*(t) + (1+\rho(t))\left(e^{i\theta_1(t)}(i\theta_1'(t)\cos\alpha(t) - \alpha'(t)\sin\alpha(t)), \zeta(t)\alpha'(t)\cos\alpha(t) + \zeta'(t)\sin\alpha(t)\right) \end{aligned}$$

so that

$$|\gamma^{*\prime}(t)|^2 = |\theta_1'(t)|^2 \cos^2\alpha(t) + |\alpha'(t)|^2 + |\zeta'(t)|^2 \sin^2\alpha(t),$$

$$|\gamma'(t)|^2 = \rho'(t)^2 + (1+\rho(t))^2 \left(|\theta_1'(t)|^2 \cos^2\alpha(t) + |\alpha'(t)|^2 + |\zeta'(t)|^2 \sin^2\alpha(t)\right),$$

and

$$\begin{aligned} \gamma'(t)_{\gamma(t)} &= \rho'(t) + (1+\rho(t))\langle\gamma^{*\prime}(t), \gamma^*(t)\rangle \\ &= \rho'(t) + i(1+\rho(t))\left(\theta_1'(t)\cos^2\alpha(t) - \operatorname{Im}(\langle\zeta(t), \zeta'(t)\rangle)\sin^2\alpha(t)\right), \end{aligned}$$

$$|\gamma'(t)_{\gamma(t)}|^2 = \rho'(t)^2 + (1+\rho(t))^2 \left|\theta_1'(t)\cos^2\alpha(t) - \operatorname{Im}(\langle\zeta(t), \zeta'(t)\rangle)\sin^2\alpha(t)\right|^2.$$

Because of (21),

$$(25)\quad k_D(z,w) \gtrsim$$
$$\int_0^1 \left(\frac{1}{\sqrt{\rho(t)}} \left|\theta_1'(t)\cos^2\alpha(t) - \mathrm{Im}(\langle \zeta(t), \zeta'(t)\rangle)\sin^2\alpha(t)\right| + |\theta_1'(t)\cos\alpha(t)| + |\zeta'(t)\sin\alpha(t)| \right) dt$$
$$+ \int_0^1 \left(|\alpha'(t)| + \frac{|\rho'(t)|}{\sqrt{\rho(t)}} \right) dt.$$

The second integral is comparable to the total variation of $|\alpha| + \sqrt{\rho}$, so larger than $\beta + \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}$. If the total variation is larger than $c_1 H_D(z,w)$ ($c_1 > 0$ to be chosen), we are done, so assume it is smaller. Since $\int_0^1 |\alpha'(t)| dt \geq \beta$, then from (23)

$$\beta \leq c_1 C_2 \left(\beta + \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}} \right) \right).$$

Choosing c_1 so that $c_1 C_2 \leq \frac{1}{2}$, we get $\beta \leq 2c_1 C_2 \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}}\right)$. Therefore re-applying (23), we have

$$H_D(z,w) \leq C_2(1+2c_1) \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}} \right). \tag{26}$$

Finally, since the total variation of $|\alpha| + \sqrt{\rho}$ is bounded by $k_D(z,w) \lesssim H_D(z,w)$, we can choose c_1 so that

$$\max_{t\in[0;1]} (|\alpha(t)| + \sqrt{\rho(t)}) \leq c_1 H_D(z,w) \leq \frac{1}{2} \min\left(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon_1}} \right) \leq \frac{1}{2}. \tag{27}$$

We want to distinguish between cases according to how "big" is the set where the first term in the integrand is relatively big with respect to an appropriate measure. More explicitly, let

$$A := \left\{ t \in [0;1] : \left|\theta_1'(t)\cos^2\alpha(t) - \mathrm{Im}(\langle \zeta(t), \zeta'(t)\rangle)\sin^2\alpha(t)\right| \geq \frac{\sqrt{\rho(t)}}{\max(\sqrt{|\eta|}, \sqrt{\varepsilon_1})} |\theta_1'(t)| \right\},$$

then let

$$\mu := \frac{\int_A |\theta_1'(t)|\, dt}{\int_0^1 |\theta_1'(t)|\, dt}.$$

Case 1: $\mu \geq \frac{1}{2}$.
Then

$$\int_0^1 \frac{1}{\sqrt{\rho(t)}} \left|\theta_1'(t)\cos^2\alpha(t) - \mathrm{Im}(\langle \zeta(t), \zeta'(t)\rangle)\sin^2\alpha(t)\right|$$
$$\geq \frac{1}{\max(\sqrt{|\eta|}, \sqrt{\varepsilon})} \int_A |\theta_1'(t)|\, dt \geq \frac{1}{2\max(\sqrt{|\eta|}, \sqrt{\varepsilon})} \int_0^1 |\theta_1'(t)|\, dt$$
$$\geq \frac{\eta}{2\max(\sqrt{|\eta|}, \sqrt{\varepsilon})} \gtrsim \min(\sqrt{|\eta|}, \frac{|\eta|}{\sqrt{\varepsilon}}).$$

Case 2: $\mu < \frac{1}{2}$.
Then, writing $A^c := [0;1] \setminus A$, $\int_{A^c} |\theta_1'(t)|\, dt \geq \frac{1}{2}\int_0^1 |\theta_1'(t)|\, dt$. When $t \in A^c$,

$$|\zeta'(t)| \sin^2\alpha(t) \geq \left(\cos^2\alpha(t) - \frac{\sqrt{\rho(t)}}{\max(\sqrt{|\eta|}, \sqrt{\varepsilon})} \right) |\theta_1'(t)| \geq \frac{1}{4}|\theta_1'(t)|$$

because of (27). Thus

$$\int_0^1 |\zeta'(t)||\sin\alpha(t)|\,dt \geq \frac{1}{\max_{t\in[0;1]}|\sin\alpha(t)|}\frac{1}{2}\int_{A^c}|\theta_1'(t)|\,dt \gtrsim \frac{1}{\sqrt{\eta}}\int_0^1 |\theta_1'(t)|\,dt \gtrsim \sqrt{\eta},$$

and we are done. □

6. Refinements of Theorems 5 and 6

Note that if p is a strongly pseudoconvex point of a domain D in $\mathbb{C}^n$, then

$$(28)\quad (1-C\sqrt{\delta_D(z)})^2\frac{|X_z|^2}{4\delta_D^2(z)}+\frac{C^{-1}|X|^2}{\delta_D(z)} \leq \kappa_D^2(z;X) \leq (1+C\sqrt{\delta_D(z)})^2\frac{|X_z|^2}{4\delta_D(z)}+\frac{C|X|^2}{\delta_D(z)},$$

where $C>1$ and $z\in D$ near p (see e.g. [1, Proposition 1.2]).

So, similarly to [1, Theorem 1.1], it is natural to extend Theorems 5 and 6 for large classes of Finsler metrics.

Let $\psi:\mathbb{R}^+\to\mathbb{R}$ be such a function that $\int_0^1\frac{|\psi(x)|}{x}dx$ exists, and let $C>0$. For a $\mathcal{C}^{1,1}$-smooth boundary point p of a domain D in $\mathbb{C}^n$, consider a Finsler pseudometrics $\mathcal{F}_D: D\times\mathbb{C}^n\to\mathbb{R}_0^+$ such that for $z\in D$ near p,

$$\mathcal{F}_D(z;X)=(1+\psi(\delta_D(z)))\frac{|X_z|}{2\delta_D(z)}+\frac{C|X|}{\sqrt{\delta_D(z)}}.$$

Denote by f_D the integrated form of $\mathcal{F}_D$.

Proposition 18. *There exists $C'>0$ (depending only on C and $\chi_{D,p}$) such that*

$$f_D(z,w)\leq\log(1+C'A_D(z,w)),\quad z,w\in D \text{ near } p.$$

The proof follows similar lines to that of Theorem 6. The only point we have to mention is that the integrability of $\psi(x)/x$ implies that if $z,z'\in D$ near p with $\pi(z)=\pi(z')$, then

$$f_D(z,z')\leq\left|\log\frac{\sqrt{\delta_D(z')}}{\sqrt{\delta_D(z)}}\right|+o(1)$$

(by integrating $\mathcal{F}_D$ on the segment $[z,z']$).

Note that Proposition 18 has an obvious analogue for Finsler metrics in a domain of $\mathbb{R}^n$ which satisfy the hypothesis of Proposition 18 with the Hermitian product replaced by the ordinary scalar product.

Let now $\mathcal{G}_D: D\times\mathbb{C}^n\to\mathbb{R}_0^+$ be a Finsler pseudometric such that for $z\in D$ near p,

$$\mathcal{G}_D(z;X)=\max\Big\{(1+\psi(\delta_D(z)))\frac{|X_z|}{2\delta_D(z)},\frac{C|X|}{\sqrt{\delta_D(z)}}\Big\}.$$

Denote by g_D the integrated form of $\mathcal{G}_D$.

Proposition 19. *If p is strongly pseudoconvex, then there exists $c'>0$ such that*

$$g_D(z,w)\geq\log(1+c'A_D(z,w)),\quad z,w\in D \text{ near } p.$$

Proof. Having in mind (3), in the case where $\psi(t)=Ct^s$, $s>0$, [1, Theorem 1.1] shows that $g_D(z,w)\geq g(z,w)-C''$, where g is a quantity which by (3) can be seen to satisfy $g(z,w)=\log(1+A_D(z,w))+O(1)$. This implies the bound we want when $A_D(z,w)\gtrsim 1$. It remains to see that we can extend this to our general case. The only place where the special form of ψ is used is [1, Lemma 4.1], and retracing the computations in the proof of that Lemma [1, p. 517] we find

$$g_D(z,w)\geq\left|\log\frac{\sqrt{\delta_D(z')}}{\sqrt{\delta_D(z)}}\right|-\int_0\frac{|\psi(t^2)|}{t}dt,$$

and this last integral converges by our hypothesis on ψ.

Finally, by (28) $\mathcal{G}_D \asymp \kappa_D$. Hence (5) implies that $g_D(z,w) \gtrsim A_D(z,w)$ if $A_D(z,w) \lesssim 1$. □

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N. Nikolov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev 8, 1113 Sofia, Bulgaria

Faculty of Information Sciences, State University of Library Studies and Information Technologies, Shipchenski prohod 69A, 1574 Sofia, Bulgaria

Email address: `nik@math.bas.bg`

P.J. Thomas, Institut de Mathématiques de Toulouse; UMR5219, Université de Toulouse; CNRS, UPS, F-31062 Toulouse Cedex 9, France

Email address: `pascal.thomas@math.univ-toulouse.fr`