

BASIC ASYMPTOTIC INVARIANTS FOR TIME-LIKE SURFACES WITH REAL ASYMPTOTIC LINES IN $\mathbb{R}_1^3$

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ABSTRACT. The geometrically defined wide class of time-like surfaces in $\mathbb{R}_1^3$ admitting real asymptotic lines is considered. A fundamental theorem of Bonnet-type is obtained for these surfaces. It states that a surface in this class is determined (up to a motion) by four invariant functions, satisfying some natural PDEs. Then canonical parameters are defined for these surfaces and it is proved that such a surface is determined (up to a motion) in canonical parameters with only two invariant functions (which in particular can be the Gauss and the mean curvature) satisfying a partial differential equation equivalent to the Gauss equation.

1. INTRODUCTION

A fundamental problem in differential geometry is to characterize an object by functions naturally connected with the object. A classical and well known example in this kind is the characterization of a curve in the space by its curvature and torsion as functions of the natural parameter. Another example is the Bonnet's classical theorem, according to which a surface in $\mathbb{R}^3$ is determined by the coefficients of the first and the second fundamental forms satisfying the equations of Gauss and Codazzi.

In these two examples we can see a principal difference in the investigations of curves and of surfaces – in the second case we need a relation between the function determining the surface. This seems very natural according to the Gauss and Codazzi equations. Another difference in these two examples is that in the first one the curvature and the torsion of a curve are invariants of the curve, while the coefficients of the fundamental forms of a surface are not. The clarification of the role of invariants in the determination of surfaces begins in [1], where Bonnet studies the relationship between the surface and its first fundamental form and principal curvatures. For other works in this direction see e.g. É. Cartan [2], S.-S. Chern [3], F. Rellich [13], W. Scherrer [14].

Recently many works have investigated the problem of determining the surfaces of special types in the 3- or 4-dimensional Euclidean space as well as in pseudo-Euclidean spaces by invariants. G. Ganchev and V. Mihova [6] proved that a Weingarten surface in $\mathbb{R}^3$ is determined up to a motion in special parameters by three functions (one of which is invariant and the two other determine the Weingarten nature of the surface) which satisfy a natural PDE. By the same arguments a similar result holds for space-like Weingarten surfaces as well as for time-like Weingarten surfaces in the Lorentz space $\mathbb{R}_1^3$, if the Gauss curvature K and the mean curvature H satisfy $K - H^2 < 0$, see [11]. R. Tribucy and I. Guadalupe [15] proved that a minimal surface in $\mathbb{R}^4$ is determined up to a motion by the

Key words and phrases. Time-like surface, Asymptotic parameters, Canonical parameters, Basic asymptotic invariants.

2020 Mathematics Subject Classification: Primary 53A35; Secondary 53B30.

Gauss and the normal curvature satisfying a system of two PDEs. Similar problems for minimal surfaces in $\mathbb{R}_1^4$ and $\mathbb{R}_2^4$ are considered respectively in [7] and [5].

In the case of an arbitrary non-umbilical surface in $\mathbb{R}^3$, the present author proved that the surface admits special principal parameters and the surface is uniquely determined (up to a motion) in these parameters by the Gauss and the mean curvature functions satisfying a PDE, see [12]. The same question is considered in [10] for surfaces with non-vanishing $K - H^2$ in $\mathbb{R}_1^3$. The arguments and the results use isotropic (or null) parameters and the main result contains an integro-differential equation, that is not usual and not very convenient. On the other hand, the considerations of space-like surfaces and of time-like surfaces in the $\mathbb{R}_1^3$ with $K - H^2 < 0$ are absolutely similar to the case of surfaces in $\mathbb{R}^3$. So only the case of time-like surfaces satisfying $K - H^2 > 0$ (so, admitting real asymptotic lines) is considerably different and deserves a special interest.

Exactly this case is the object of the present paper. It turns out that the asymptotic parameters are very useful in our case. We prove that the surface is uniquely determined in asymptotic parameters by four invariant geometrically defined functions, satisfying a system of three PDSs. Then we show that the surface admits special asymptotic parameters (we call them canonical asymptotic parameters). In the particular case, when the surface is minimal our definition of canonical asymptotic parameters coincides with the definition of canonical parameters given in [4]. We obtain a fundamental theorem, stating that a time-like surface in $\mathbb{R}_1^3$ with $K - H^2 > 0$ is determined in canonical asymptotic parameters up to a motion in the space by its Gauss and mean curvature functions, satisfying a PDE, equivalent to the Gauss equation. When the Gauss curvature is constant this equation is studied in [9].

2. PRELIMINARIES

Let $\mathbb{R}_1^3$ be the 3-dimensional pseudo-Euclidean space with scalar product given by

$$\langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3 .$$

A vector x is called space-like, time-like or light-like if $\langle x, x \rangle$ is > 0 , < 0 or $= 0$, respectively. A surface S in $\mathbb{R}_1^3$ is said to be space-like or time-like if at any point of S its normal vector is time-like, resp. space-like. Then the induced metric on S is respectively Riemannian or Lorentzian.

Suppose the time-like surface S is defined by $z = z(u, v)$ and denote the derivatives of $z(u, v)$ by z_u, z_v, z_{uu} , etc. The coefficients of the first fundamental form are given by

$$E = z_u^2 \quad F = \langle z_u, z_v \rangle \quad G = z_v^2 .$$

We denote by n the unit vector, collinear to the Lorentzian cross product $z_u \times z_v$, so that the triple $\{z_u, z_v, n\}$ is positively oriented. Then the coefficients of the second fundamental form are given by

$$L = \langle n, z_{uu} \rangle \quad M = \langle n, z_{uv} \rangle \quad N = \langle n, z_{vv} \rangle .$$

The Gauss curvature K and the mean curvature H of S are defined by

$$K = \frac{LN - M^2}{EG - F^2} \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)} .$$

When $E = G = 0$, the parameters (u, v) are called isotropic or null parameters. For them the parametric lines are isotropic. In [10] the geometry of a time-like surface is considered using isotropic parameters. On the other hand, in the study of space-like surfaces or time-like surfaces with $K - H^2 < 0$, the principal parameters are more natural and more useful. For these parameters the parametric lines are principal and they are characterized by $F = M = 0$. Then the theory of the surface can be developed analogously to that in the classical Riemannian geometry, see [11]. In our study we shall use asymptotic parameters, i.e. these for which the parametric lines are asymptotic. Analytically they are characterized by $L = N = 0$. Two pairs of asymptotic parameters (u, v) and $(\bar{u}, \bar{v})$ in a neighbourhood of a point are related either by $u = u(\bar{u})$, $v = v(\bar{v})$, or $u = u(\bar{v})$, $v = v(\bar{u})$.

It is easy to see that if the Gauss curvature K is positive, the time-like surface admits real asymptotic parameters. Suppose moreover that $K - H^2$ never vanishes. As we said, in the case $K - H^2 < 0$ the surface can be parametrized by (real) principal parameters and the theory is analogous to that in the Riemannian case. So suppose $K - H^2 > 0$, i.e. the principal curvatures are imaginary. Then in asymptotic parameters

$$K - H^2 = -EG \frac{M^2}{(EG - F^2)^2}.$$

Without loss of generality we assume $E > 0$, $G < 0$. We put

$$x = \frac{z_u}{\sqrt{E}} \quad y = \frac{z_v}{\sqrt{-G}} \quad a = \langle x, y \rangle.$$

Then $x^2 = 1$, $y^2 = -1$. Denoting by ∇ the covariant differentiation on $\mathbb{R}_1^3$ we obtain the following Frenet-type formulas:

$$\begin{aligned} \nabla_x x &= -a\gamma_1 x + \gamma_1 y \\ \nabla_x y &= \left(\frac{x(a)}{1+a^2} + \gamma_1 \right) x + a \left(\frac{x(a)}{1+a^2} + \gamma_1 \right) y + \alpha n \\ \nabla_x n &= -\frac{a\alpha}{1+a^2} x + \frac{\alpha}{1+a^2} y \\ \nabla_y x &= a \left(\frac{y(a)}{1+a^2} - \gamma_2 \right) x - \left(\frac{y(a)}{1+a^2} - \gamma_2 \right) y + \alpha n \\ \nabla_y y &= \gamma_2 x + a\gamma_2 y \\ \nabla_y n &= -\frac{\alpha}{1+a^2} x - \frac{a\alpha}{1+a^2} y \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \gamma_1 &= -\frac{1}{1+a^2} \left(x(a) + a \frac{(\sqrt{-G})_u}{\sqrt{E}\sqrt{-G}} - \frac{(\sqrt{E})_v}{\sqrt{E}\sqrt{-G}} \right) \\ \gamma_2 &= \frac{1}{1+a^2} \left(y(a) + a \frac{(\sqrt{E})_v}{\sqrt{E}\sqrt{-G}} + \frac{(\sqrt{-G})_u}{\sqrt{E}\sqrt{-G}} \right) \\ a &= \frac{F}{\sqrt{E}\sqrt{-G}} \quad \alpha = \frac{M}{\sqrt{E}\sqrt{-G}}. \end{aligned} \tag{2.2}$$

Hence we obtain also

$$K = \frac{\alpha^2}{1+a^2} \quad H = \frac{a\alpha}{1+a^2}. \tag{2.3}$$

Remark 2.1. Note that the functions a , γ_i and α are invariants of the surface. But regardless of the notations γ_i and α are not in general the geodesic curvatures and torsion of the asymptotic lines. More precisely if $\bar{\gamma}_i$ are the geodesic curvatures of the asymptotic lines, then

$$\bar{\gamma}_i = \gamma_i \sqrt{1 + a^2}.$$

Similarly, if $\bar{\alpha}$ is the torsion of the asymptotic lines, then

$$\alpha = \bar{\alpha} \sqrt{1 + a^2}.$$

In particular, if $a = 0$, i.e. if the surface is minimal, then $\bar{\gamma}_i = \gamma_i$ and $\bar{\alpha} = \alpha$.

We shall call γ_1 , γ_2 , a and α *the basic asymptotic invariants* of the surface.

Using (2.1) we derive the main equations of the surface - the Gauss equation

$$(2.4) \quad \begin{aligned} & x(\gamma_2) - y(\gamma_1) - 2a\gamma_1\gamma_2 - \gamma_1^2 + \gamma_2^2 - \gamma_1 \frac{x(a)}{1+a^2} - \gamma_2 \frac{y(a)}{1+a^2} \\ & - \frac{a_{uv}}{(1+a^2)\sqrt{E}\sqrt{-G}} + \frac{ax(a)y(a)}{(1+a^2)^2} + \frac{\alpha^2}{1+a^2} = 0 \end{aligned}$$

and the equations of Codazzi:

$$(2.5) \quad \begin{aligned} x(\alpha) &= a\alpha \frac{x(a)}{1+a^2} + 2\alpha \left(\frac{y(a)}{1+a^2} - \gamma_2 \right), \\ y(\alpha) &= a\alpha \frac{y(a)}{1+a^2} - 2\alpha \left(\frac{x(a)}{1+a^2} + \gamma_1 \right). \end{aligned}$$

Note that the expression $\frac{\alpha^2}{1+a^2}$ in the Gauss equation is exactly the Gauss curvature K .

3. DETERMINING A TIME-LIKE SURFACE BY INVARIANTS

Using (2.2), from the Codazzi equations (2.5) we find

$$(3.1) \quad \begin{aligned} \left(\sqrt{E} \right)_v &= f_v \sqrt{E} + a f_u \sqrt{-G} \\ \left(\sqrt{-G} \right)_u &= -a f_v \sqrt{E} + f_u \sqrt{-G} \end{aligned}$$

where

$$f = \frac{1}{2} \left(\log \sqrt{1+a^2} - \log |\alpha| \right).$$

Remark 3.1. Since $K = \frac{\alpha^2}{1+a^2}$ we have also $f = -\log \sqrt[4]{K}$.

Now the formulas for γ_1 and γ_2 take the form

$$(3.2) \quad \begin{aligned} \gamma_1 &= -\frac{a_u}{(1+a^2)\sqrt{E}} + \frac{f_v}{\sqrt{-G}}, \\ \gamma_2 &= \frac{a_v}{(1+a^2)\sqrt{-G}} + \frac{f_u}{\sqrt{E}}. \end{aligned}$$

From this system for $\sqrt{E}$ and $\sqrt{-G}$ we obtain

$$(3.3) \quad \begin{aligned} \sqrt{E} &= \frac{a_u a_v + (1 + a^2)^2 f_u f_v}{(1 + a^2)(-a_v \gamma_1 + (1 + a^2) f_v \gamma_2)} , \\ \sqrt{-G} &= \frac{a_u a_v + (1 + a^2)^2 f_u f_v}{(1 + a^2)(a_u \gamma_2 + (1 + a^2) f_u \gamma_1)} . \end{aligned}$$

In particular (3.3) implies that the functions on the right are positive.

The Frenet-type formulas (2.1) can be written as

$$\begin{aligned} x_u &= \sqrt{E} \gamma_1 (-ax + y) \\ y_u &= \sqrt{E} \left[\left(\frac{x(a)}{1 + a^2} + \gamma_1 \right) (x + ay) + \alpha n \right] \\ n_u &= \sqrt{E} \frac{\alpha}{1 + a^2} (-ax + y) \\ x_v &= \sqrt{-G} \left[\left(\frac{y(a)}{1 + a^2} - \gamma_2 \right) (ax - y) + \alpha n \right] \\ y_v &= \sqrt{-G} \gamma_2 (x + ay) \\ n_v &= -\sqrt{-G} \frac{\alpha}{1 + a^2} (x + ay) . \end{aligned}$$

Using this we shall prove the following theorem

Theorem 3.1. *Assume the functions $\gamma_1(u, v)$, $\gamma_2(u, v)$, $a(u, v)$, $\alpha(u, v)$ are smooth in a neighbourhood D of a point $(u_0, v_0) \in \mathbb{R}^2$. Suppose the functions $f = \frac{1}{2} (\log \sqrt{1 + a^2} - \log |\alpha|)$,*

$$\Phi = \frac{a_u a_v + (1 + a^2)^2 f_u f_v}{(1 + a^2)(-a_v \gamma_1 + (1 + a^2) f_v \gamma_2)} \quad \Psi = \frac{a_u a_v + (1 + a^2)^2 f_u f_v}{(1 + a^2)(a_u \gamma_2 + (1 + a^2) f_u \gamma_1)}$$

are well defined and $\Phi > 0$, $\Psi > 0$ in D . If the following conditions are satisfied:

$$(3.4) \quad \begin{aligned} &\frac{(\gamma_2)_u}{\Phi} - \frac{(\gamma_1)_v}{\Psi} - 2a\gamma_1\gamma_2 - \gamma_1^2 + \gamma_2^2 - \gamma_1 \frac{a_u}{(1 + a^2)\Phi} - \gamma_2 \frac{a_v}{(1 + a^2)\Psi} \\ &- \frac{a_{uv}}{(1 + a^2)\Phi\Psi} + \frac{aa_u a_v}{(1 + a^2)^2 \Phi\Psi} + \frac{\alpha^2}{1 + a^2} = 0 , \end{aligned}$$

$$(3.5) \quad \begin{aligned} &(\log \Phi)_v + a f_u \frac{a_v \gamma_1 - (1 + a^2) f_v \gamma_2}{a_u \gamma_2 + (1 + a^2) f_u \gamma_1} = f_v , \\ &(\log \Psi)_u + a f_v \frac{a_u \gamma_2 + (1 + a^2) f_u \gamma_1}{-a_v \gamma_1 + (1 + a^2) f_v \gamma_2} = f_u , \end{aligned}$$

then there exists a unique (up to a motion) surface in $\mathbb{R}_1^3$ with basic asymptotic invariants the given functions γ_1 , γ_2 , a , α . Moreover (u, v) are asymptotic parameters.

Proof. We shall search for unit vector fields $x(u, v)$, $y(u, v)$, $n(u, v)$, subjects to the system

$$\begin{aligned}
 x_u &= \gamma_1 \Phi(-ax + y) \\
 y_u &= \left(\frac{a_u}{1+a^2} + \gamma_1 \Phi \right) (x + ay) + \alpha \Phi n \\
 n_u &= \frac{\alpha \Phi}{1+a^2} (-ax + y) \\
 x_v &= \left(\frac{a_v}{1+a^2} - \gamma_2 \Psi \right) (ax - y) + \alpha \Psi n \\
 y_v &= \gamma_2 \Psi (x + ay) \\
 n_v &= -\frac{\alpha \Psi}{1+a^2} (x + ay)
 \end{aligned}
 \tag{3.6}$$

or briefly

$$\xi_u = U\xi \quad \xi_v = V\xi ,$$

where $\xi = (x, y, n)^t$,

$$\begin{aligned}
 U &= \Phi \begin{pmatrix} \frac{-a\gamma_1}{\frac{a_u}{(1+a^2)\Phi} + \gamma_1} & \frac{aa_u}{(1+a^2)\Phi} + a\gamma_1 & 0 \\ \frac{a\alpha}{1+a^2} & \frac{\alpha}{1+a^2} & \alpha \\ -\frac{1}{1+a^2} & \frac{1}{1+a^2} & 0 \end{pmatrix} , \\
 V &= \Psi \begin{pmatrix} \frac{aa_v}{(1+a^2)\Psi} - a\gamma_2 & \frac{-a_v}{(1+a^2)\Psi} + \gamma_2 & \alpha \\ \frac{\gamma_2}{\alpha} & \frac{a\gamma_2}{a\alpha} & 0 \\ -\frac{1}{1+a^2} & -\frac{1}{1+a^2} & 0 \end{pmatrix} .
 \end{aligned}$$

The integrability condition of this system is

$$U_v - V_u = VU - UV .$$

Using the condition of the theorem we can conclude that this condition is fulfilled. Consequently there exist unique vector fields x , y , n , defined in a subset D_0 of D such that

$$x(u_0, v_0) = x_0 \quad y(u_0, v_0) = y_0 \quad n(u_0, v_0) = n_0 ,$$

where (u_0, v_0) is a point in D and $\{x_0, y_0, n_0\}$ is a positively oriented orthonormal basis in a point z_0 of $\mathbb{R}_1^3$, such that

$$x_0^2 = 1 \quad x_0 y_0 = a(u_0, v_0) \quad y_0^2 = -1 \quad n_0^2 = 1 \quad x_0 n_0 = 0 \quad y_0 n_0 = 0 .$$

A standard procedure leads to the conclusion that

$$x^2 = 1 \quad xy = a \quad y^2 = -1 \quad xn = 0 \quad yn = 0 \quad n^2 = 1$$

because this is true at (u_0, v_0) . The triple $\{x, y, n\}$ is positively oriented, because this is so at (u_0, v_0) .

Now we look for a vector valued function $z(u, v)$, such that

$$z_u = \Phi x \quad z_v = \Psi y .$$

The integrability condition

$$(\Phi x)_v = (\Psi y)_u$$

is fulfilled and hence there exists a unique function $z(u, v)$, defined in a sub-domain of D_0 , such that $z(u_0, v_0) = z_0$.

On the other hand, as x, y, n satisfy (3.6), then (u, v) are asymptotic parameters and the given functions $\gamma_1, \gamma_2, a, \alpha$ are the basic asymptotic invariants of the surface. $\square$

4. CANONICAL PARAMETERS FOR SURFACES WITH POSITIVE GAUSS CURVATURE

Formulas (3.1) imply easily that for any point (u_0, v_0) the functions

$$(4.1) \quad \begin{aligned} \sqrt{E}e^{-\int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) dv - \int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) (u, v_0) du - f(u_0, v_0)} \\ \sqrt{-G}e^{-\int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) du - \int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) (u_0, v) dv - f(u_0, v_0)} \end{aligned},$$

do not depend on v and u , respectively. To work with the most natural simplification of these functions, we give the following definition:

Definition 4.1. We say that the asymptotic parameters (u, v) of a surface of positive Gauss curvature in $\mathbb{R}_1^3$ are *canonical asymptotic parameters* if the functions $\varphi(u)$ and $\psi(v)$ defined by (4.1) are equal to 1.

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In other words the asymptotic parameters are canonical if

$$\begin{aligned} \sqrt{E} &= e^{\int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) dv + \int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) (u, v_0) du + f(u_0, v_0)} \\ \sqrt{-G} &= e^{\int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) du + \int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) (u_0, v) dv + f(u_0, v_0)}. \end{aligned}$$

Remark 4.1. In general the definition depends on the point (u_0, v_0) , so it is more exact to speak about *canonical asymptotic parameters about a fixed point*.

Proposition 4.2. *Let S be a surface with positive Gauss curvature in $\mathbb{R}_1^3$. Then locally S admits canonical asymptotic parameters.*

Proof. Taking arbitrary asymptotic parameters (u, v) we define new parameters $\bar{u}, \bar{v}$ by

$$\bar{u} = \int_{u_0}^u \varphi(u) + u_0, \quad \bar{v} = \int_{v_0}^v \psi(v) + v_0.$$

Then $\bar{u} = \bar{u}(u)$ and $\bar{v} = \bar{v}(v)$, so the new parameters $\bar{u}, \bar{v}$ are also asymptotic. On the other hand

$$\bar{u}_u = \sqrt{E}e^{-\int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) dv - \int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) (u, v_0) du - f(u_0, v_0)},$$

$$\bar{v}_v = \sqrt{-G}e - \int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{G}} + f_u \right) du - \int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) (u_0, v) dv - f(u_0, v_0)$$

imply

$$\bar{E} = e \left(2 \left(\int_{v_0}^v \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) dv + \int_{u_0}^u \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) (u, v_0) du + f(u_0, v_0) \right) \right).$$

Moreover a straightforward verification gives

$$\begin{aligned} \left(f_v + af_u \frac{\sqrt{-G}}{\sqrt{E}} \right) (u, v) &= \left(\bar{f}_{\bar{v}} + \bar{a} \bar{f}_{\bar{u}} \frac{\sqrt{-\bar{G}}}{\sqrt{\bar{E}}} \right) (\bar{u}(u), \bar{v}(v)) \frac{d\bar{v}}{dv}, \\ \left(-af_v \frac{\sqrt{E}}{\sqrt{-G}} + f_u \right) (u, v) &= \left(-\bar{a} \bar{f}_{\bar{v}} \frac{\sqrt{\bar{E}}}{\sqrt{-\bar{G}}} + \bar{f}_{\bar{u}} \right) (\bar{u}(u), \bar{v}(v)) \frac{d\bar{u}}{du}. \end{aligned}$$

Denote $\bar{u}_0 = \bar{u}(u_0) = u_0$, $\bar{v}_0 = \bar{v}(v_0) = v_0$. Hence we derive

$$\bar{E} = e \left(2 \left(\int_{\bar{v}_0}^{\bar{v}} \left(\bar{f}_{\bar{v}} + \bar{a} \bar{f}_{\bar{u}} \frac{\sqrt{-\bar{G}}}{\sqrt{\bar{E}}} \right) (\bar{u}, \bar{v}) d\bar{v} + \int_{\bar{u}_0}^{\bar{u}} \left(-\bar{a} \bar{f}_{\bar{v}} \frac{\sqrt{\bar{E}}}{\sqrt{-\bar{G}}} + \bar{f}_{\bar{u}} \right) (\bar{u}, \bar{v}_0) d\bar{u} + f(\bar{u}_0, \bar{v}_0) \right) \right)$$

i.e. $\bar{\varphi}(\bar{u}) = 1$. Analogously, $\bar{\psi}(\bar{v}) = 1$. So, the parameters $(\bar{u}, \bar{v})$ are canonical. $\square$

The following assertion gives the connection between different pairs of canonical asymptotic parameters.

Lemma 4.3. *If (u, v) and $(\bar{u}, \bar{v})$ are canonical asymptotic parameters in a neighbourhood of a point p , then*

$$\begin{cases} \bar{u} = \pm u + u_1 \\ \bar{v} = \pm v + v_1 \end{cases} \quad \text{or} \quad \begin{cases} \bar{u} = \pm v + v_1 \\ \bar{v} = \pm u + u_1 \end{cases}$$

for some constants u_1, v_1 .

Assume now that S is parametrized by canonical asymptotic parameters. Then the Gauss equation (2.4), can be written in the form

$$\begin{aligned} & \frac{1}{\sqrt{E}\sqrt{-G}} \left(\frac{a_{uv}}{1+a^2} - \frac{aa_u a_v}{(1+a^2)^2} - 2af_u f_v \right) + \frac{aa_u f_u}{(1+a^2)E} + \frac{aa_v f_v}{(1+a^2)G} \\ & + \frac{(f_u)^2}{E} + \frac{(f_v)^2}{G} + \frac{1}{\sqrt{E}} \left(\frac{f_u}{\sqrt{E}} \right)_u - \frac{1}{\sqrt{-G}} \left(\frac{f_v}{\sqrt{-G}} \right)_v + \frac{\alpha^2}{1+a^2} = 0. \end{aligned}$$

Hence we can prove the following theorem:

Theorem 4.4. *Assume the functions $a(u, v)$, $\alpha(u, v)$ are smooth in a neighbourhood D of a point $(u_0, v_0) \in \mathbb{R}^2$, $\alpha > 0$ in D . Define $f = \frac{1}{2} (\log \sqrt{1+a^2} - \log \alpha)$. Let (Φ, Ψ) be the solution of the PDE system*

$$(4.2) \quad \begin{cases} \Phi_v = f_v \Phi + af_u \Psi \\ \Psi_u = -af_v \Phi + f_u \Psi \end{cases}$$

with initial conditions

$$\begin{aligned}\Phi(u, v_0) &= e^{\int_{u_0}^u (-af_v + f_u)(u, v_0)du} + f(u_0, v_0), \\ \Psi(u_0, v) &= e^{\int_{v_0}^v (f_v + af_u)(u_0, v)dv} + f(u_0, v_0).\end{aligned}$$

If the equation

$$(4.3) \quad \begin{aligned} & \frac{1}{\Phi\Psi} \left(\frac{a_{uv}}{1+a^2} - \frac{aa_u a_v}{(1+a^2)^2} - 2af_u f_v \right) + \frac{aa_u f_u}{(1+a^2)\Phi^2} + \frac{aa_v f_v}{(1+a^2)\Psi^2} \\ & + \frac{(f_u)^2}{\Phi^2} + \frac{(f_v)^2}{\Psi^2} + \frac{1}{\Phi} \left(\frac{f_u}{\Phi} \right)_u - \frac{1}{\Psi} \left(\frac{f_v}{\Psi} \right)_v + \frac{\alpha^2}{1+a^2} = 0 \end{aligned}$$

is satisfied, then there exists a unique (up to a motion) surface in $\mathbb{R}_1^3$ with basic asymptotic invariants the given functions a, α . Moreover (u, v) are canonical asymptotic parameters.

Proof. We introduce the functions

$$(4.4) \quad \gamma_1 = -\frac{a_u}{(1+a^2)\Phi} + \frac{f_v}{\Psi}, \quad \gamma_2 = \frac{a_v}{(1+a^2)\Psi} + \frac{f_u}{\Phi}.$$

Using (4.2), (4.3) and (4.4) we can check that the conditions of Theorem 3.1 are fulfilled. Hence there exists a unique (up to a motion) surface in $\mathbb{R}_1^3$ with basic asymptotic invariants $\gamma_1, \gamma_2, a, \alpha$ and (u, v) are asymptotic parameters.

Finally, since the pair (Φ, Ψ) is a solution of the Cauchy problem in the condition of the theorem, then (u, v) are canonical asymptotic parameters. $\square$

To obtain the above theorem in a form with the classical invariants K and H , we first remember that the Gauss curvature K and the mean curvature H of a surface, parametrized with asymptotic parameters satisfy:

$$K = \frac{\alpha^2}{1+a^2}, \quad H = \frac{a\alpha}{1+a^2}.$$

Conversely,

$$a = \frac{H}{\sqrt{K-H^2}}, \quad \alpha = \frac{K}{\sqrt{K-H^2}}$$

or

$$a = -\frac{H}{\sqrt{K-H^2}}, \quad \alpha = -\frac{K}{\sqrt{K-H^2}}.$$

Consequently, from Theorem 4.4 we obtain:

Theorem 4.5. Assume the functions $K(u, v), H(u, v)$ are smooth in a neighbourhood D of a point $(u_0, v_0) \in \mathbb{R}^2$, $K > 0$, $K - H^2 > 0$ in U . Define $a = \frac{H}{\sqrt{K-H^2}}$. Let (Φ, Ψ) be the solution of the PDE system

$$\begin{cases} \Phi_v = -\frac{1}{4K} (K_v \Phi + aK_u \Psi) \\ \Psi_u = \frac{1}{4K} (aK_v \Phi - K_u \Psi) \end{cases}.$$

with initial conditions

$$\begin{aligned}\Phi(u, v_0) &= e^{\int_{u_0}^u \left(a(\log \sqrt[4]{K})_v - (\log \sqrt[4]{K})_u \right) (u, v_0) du} - \log \sqrt[4]{K}(u_0, v_0) , \\ \Psi(u_0, v) &= e^{\int_{v_0}^v \left(-(\log \sqrt[4]{K})_v - a(\log \sqrt[4]{K})_u \right) (u_0, v) dv} - \log \sqrt[4]{K}(u_0, v_0) .\end{aligned}$$

If the equation

$$\begin{aligned}\frac{1}{\Phi\Psi} \left(\frac{a_{uv}}{1+a^2} - \frac{aa_u a_v}{(1+a^2)^2} - \frac{aK_u K_v}{8K^2} \right) - \frac{aa_u K_u}{4(1+a^2)K\Phi^2} - \frac{aa_v K_v}{4(1+a^2)K\Psi^2} \\ + \frac{(K_u)^2}{16\Phi^2 K^2} + \frac{(K_v)^2}{16\Psi^2 K^2} - \frac{1}{4\Phi} \left(\frac{K_u}{\Phi K} \right)_u + \frac{1}{4\Psi} \left(\frac{K_v}{\Psi K} \right)_v + K = 0\end{aligned}$$

is satisfied, then there exists a unique (up to a motion) surface in $\mathbb{R}_1^3$ with Gauss curvature K and mean curvature H . Moreover (u, v) are canonical asymptotic parameters.

In particular, if S is minimal, our notations imply $a = 0$, $K = \alpha^2$, $f = -\log \sqrt{\alpha}$. Definition 4.1 of canonical asymptotic parameters is now equivalent to

$$\begin{aligned}E &= \frac{1}{\alpha} , & F &= 0 , & G &= -\frac{1}{\alpha} , \\ L &= 0 , & M^2 &= 1 , & N &= 0 .\end{aligned}$$

Consequently in this case our definition of canonical asymptotic parameters coincides with the definition of canonical parameters, given in [4].

Let now S be a surface of constant positive Gauss curvature and imaginary principal curvatures. Up to similarity we suppose that the Gauss curvature $K = 1$. Then Theorem 4.5 implies

$$\Phi_v = 0 \quad \Psi_u = 0 \quad \Phi(u, v_0) = 1 \quad \Psi(u_0, v) = 1 ,$$

so $\Phi = \Psi = 1$ and the Gauss equation takes the form

$$(4.5) \quad \frac{a_{uv}}{1+a^2} - \frac{aa_u a_v}{(1+a^2)^2} + 1 = 0$$

where

$$a = \frac{H}{\sqrt{1-H^2}} .$$

If we put $a = \sinh \omega$ in (4.5) we obtain the equation of the surface of Gauss curvature 1 in the form of a cosh-Gordon equation as in [9]:

$$\omega_{uv} + \cosh \omega = 0 .$$

In accordance with the general notion of Chebyshev coordinates (see e.g. [8]), (u, v) are called in [9] asymptotic Chebyshev coordinates.

5. EXAMPLES

Example 5.1. Consider the time-like surface in $\mathbb{R}_1^3$ with the parametrization

$$z(u, v) = \left(\frac{v^3}{6} + \frac{u^2 v}{2} - \frac{v}{2}, \frac{u^2}{2} + \frac{v^2}{2}, \frac{u^3}{6} + \frac{uv^2}{2} + \frac{v^3}{6} + \frac{u}{2} \right).$$

The coefficients of the first and the second fundamental forms are respectively

$$\begin{aligned} E &= -\frac{1}{4}(1 - u^2 + v^2)^2 & F &= 0 & G &= \frac{1}{4}(1 - u^2 + v^2)^2, \\ L &= -1 & M &= 0 & N &= 1. \end{aligned}$$

For the Gauss curvature and the mean curvature we find

$$K = -\frac{16}{(1 - u^2 + v^2)^4} < 0 \quad H = 0.$$

Hence the surface has (real) principal parameters and imaginary asymptotic parameters. As explained in Introduction, for such surfaces the canonical (principal) parameters are defined as in [11] following the procedure for surfaces in $\mathbb{R}^3$.

This surface is an Enneper-type time-like surface of negative Gauss curvature.

Example 5.2. For the time-like surface in $\mathbb{R}_1^3$ with the parametrization

$$z(u, v) = \left(\frac{u^3}{6} + \frac{uv^2}{2} - \frac{u}{2}, uv, \frac{u^2 v}{2} + \frac{v^3}{6} + \frac{v}{2} \right)$$

the coefficients of the first and the second fundamental forms are respectively

$$\begin{aligned} E &= \frac{1}{4}(1 - u^2 + v^2)^2 & F &= 0 & G &= -\frac{1}{4}(1 - u^2 + v^2)^2, \\ L &= 0 & M &= 1 & N &= 0. \end{aligned}$$

Then for the Gauss curvature and the mean curvature we obtain

$$K = \frac{16}{(1 - u^2 + v^2)^4} > 0 \quad H = 0 \quad K - H^2 = \frac{16}{(1 - u^2 + v^2)^4} > 0.$$

Hence the surface admits real asymptotic lines. Moreover, it can be checked that (u, v) are canonical asymptotic parameters.

This surface is an Enneper-type time-like surface of positive Gauss curvature.

Example 5.3. Consider the rotational surface given by

$$z(u, v) = \left(u, \cos u \cosh v, \cos u \sinh v \right)$$

for $|u| < \pi/2$. Then the coefficients of the first and the second fundamental forms are respectively

$$\begin{aligned} E &= 1 + \sin^2 u & F &= 0 & G &= -\cos^2 u, \\ L &= \frac{\cos u}{\sqrt{1 + \sin^2 u}} & M &= 0 & N &= -\frac{\cos u}{\sqrt{1 + \sin^2 u}}. \end{aligned}$$

Hence for the Gauss and the mean curvature we obtain

$$K = \frac{1}{(1 + \sin^2 u)^2} \quad H = \frac{1}{\cos u (\sqrt{1 + \sin^2 u})^3} \quad K - H^2 = -\frac{\sin^2 u \tan^2 u}{(1 + \sin^2 u)^3}.$$

Here $K - H^2 < 0$, but $K > 0$, so the surface admits both principal and asymptotic parameters. For example after the change

$$u = \sinh \bar{u} + \bar{v} \quad v = \sinh \bar{u} - \bar{v}$$

we see that the parameters $(\bar{u}, \bar{v})$ are asymptotic. But in these parameters $\bar{E} > 0$, $\bar{G} > 0$, so we can not use our constructions.

Example 5.4. Consider the Lorentz sphere with parametrization

$$z(u, v) = \left(\frac{\cosh v}{\cosh u}, \tanh u, \frac{\sinh v}{\cosh u} \right).$$

Then

$$\begin{aligned} E &= \frac{1}{\cosh^2 u} & F &= 0 & G &= -\frac{1}{\cosh^2 u}, \\ L &= -\frac{1}{\cosh^2 u} & M &= 0 & N &= \frac{1}{\cosh^2 u}. \end{aligned}$$

Consequently, $K = 1$, $K - H^2 = 1$. Trying to obtain asymptotic parameters we put

$$u = \bar{u} - \bar{v} \quad v = \bar{u} + \bar{v}.$$

Now we find that the resulting parameters are isotropic, i.e. $\bar{E} = \bar{G} = 0$. So our method is not applicable.

The last two examples show that the requirement $K - H^2 > 0$ is essential for our method.

Funding. The author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria, under contract KP-06-N82/6.

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