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Special Structures on General Rotational Surfaces in the Pseudo-Euclidean 4-space with Neutral Metric

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Dedicated to Professor Bang-Yen Chen's Anniversary

Abstract. We study general rotational surfaces in the pseudo-Euclidean 4-dimensional space with neutral metric and describe the behavior of geometric objects, such as Killing vector fields (and in particular homothetic vector fields), divergence-free vector fields, co-closed and harmonic one-forms, and also harmonic functions. We classify geodesic and parallel vector fields, geodesic curves, concircular vector fields and concircular functions, and also concurrent vector fields and functions whose gradient is concurrent. Our results are new, as they have not been obtained in the Euclidean and Minkowski framework. The tools here are taken from both differential geometry and partial and ordinary differential equations.

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1. Introduction

Rotational surfaces are a widely studied subject, being central to many fields of mathematics, physics, engineering, and architecture. An overview of this topic could be found in [1]. In our study we focus on general rotational surfaces in the pseudo-Euclidean 4-dimensional space with neutral metric. Historically, the study of these rotational surfaces was initiated at the beginning of the 20th century, in the context of 4-dimensional Euclidean spaces by C. Moore [2,3]. A special case of such surfaces was described in depth in a number of papers by the second author, as well as by U. Dursun and N.-C. Turgay [4]. Also, in the isotropic 4-space [5], a similar study could be carried out. These surfaces can also be studied in the context of Lorentz metrics and curvature homogeneity [6], or as real submanifolds in complex space forms (such as complex Euclidean space, complex projective space, or complex hyperbolic space). A detailed study of real submanifolds in complex space forms (specifically Euclidean and projective spaces) is presented in [7]. Due to the importance of the rotational surfaces in the theory of relativity, it was natural that they should be studied in the Minkowski space, and this was done by G. Ganchev and the second author [8].

A metric in the case of four-dimensional manifolds that is neither Riemannian nor Lorentz must be of neutral signature (see [9,10], for example). General rotational surfaces in pseudo-Euclidean spaces of neutral signature were studied by F. Aksoyak, Y. Yayli [11], as well as by Y. Aleksieva and the second author [12].

The aim of this study is to describe new interesting features of general rotational surfaces in pseudo-Euclidean spaces of neutral signature. The content of our paper is as follows: after some preliminaries provided in Sect. 2, we establish in Sect. 3 some necessary and sufficient conditions for a general rotational surface to be immersed in the pseudo-Euclidean space of neutral signature, and then we characterize it when it is spacelike, timelike, and lightlike. Sect. 4 is dedicated to the study of general rotational surfaces with orthogonal parametric lines. Here we provide necessary and sufficient conditions under which such a surface is parametrized by isothermal coordinates. Our main results are included in Sects. 5–8. Killing vector fields (and in particular homothetic vector fields) on both general rotational surfaces and standard rotational surfaces are described in Sects. 5 and 6, respectively. Next, in Sect. 7, on both general and standard rotational surfaces, we classify divergence-free vector fields, co-closed and harmonic one-forms, and also harmonic functions. Harmonicity, being an area of interest in both analysis and differential geometry, is also included in our study. In Sect. 8, we classify geodesic and parallel vector fields, geodesic curves, concircular vector fields and concircular functions, and also concurrent vector fields and functions whose gradient is concurrent. Our results explain the behavior of geometric objects on general rotational surfaces, immersed in the remaining case, namely pseudo-Euclidean 4-dimensional space of neutral signature. These type of results are new as they have not been obtained in the other two cases, namely Euclidean and Minkowski 4-spaces. The tools here are taken from both differential geometry and partial and ordinary differential equations. Four-dimensional manifolds play a special role in mathematics, but some results obtained here could be extended to general rotational surfaces in higher dimensions.

2. Preliminaries

As usual, $(\mathbb{E}_2^4, \langle , \rangle)$ denotes the pseudo-Euclidean 4-space of signature $(+,+,-,-)$ endowed with the canonical pseudo-Euclidean neutral metric given in a rectangular coordinate system (x_1, x_2, x_3, x_4) by

$$\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2.$$

A vector $v \in \mathbb{E}_2^4$ can have one of the following three casual characters: *spacelike*, if $\langle v, v\rangle > 0$ or $v = 0$, *timelike* if $\langle v, v\rangle < 0$, and *lightlike* if $\langle v, v\rangle = 0$ and $v \neq 0$. This terminology is inspired by general relativity and the Minkowski 4-space $\mathbb{E}_1^4$.

A surface $\mathcal{M}$ in $\mathbb{E}_2^4$ is called *spacelike* (resp. *timelike* or *Lorentz*) if $\langle , \rangle$ induces a Riemannian (resp. Lorentzian) metric on $\mathcal{M}$. Thus, the orthogonal decomposition into tangent and normal space

$$\mathbb{E}_2^4 = T_p\mathcal{M} \oplus N_p\mathcal{M}$$

holds at each point $p \in \mathcal{M}$ and the metric $\langle , \rangle$ restricted onto the tangent space $T_p\mathcal{M}$ is of signature $(2, 0)$ (resp. $(1, 1)$).

If we denote by ∇ and $\widetilde{\nabla}$ the Levi Civita connections of $\mathcal{M}$ and $\mathbb{E}_2^4$, respectively, then for any vector fields X, Y tangent to $\mathcal{M}$ and a vector field ξ normal to $\mathcal{M}$, the formulas of Gauss and Weingarten are given respectively by:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y);$$
$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where σ denotes the second fundamental form, D is the normal connection, and A_ξ is the shape operator with respect to ξ. In the non-Riemannian case, the operator A_ξ may not be diagonalizable.

The mean curvature vector field H of $\mathcal{M}$ in $\mathbb{E}_2^4$ is defined as $H = \dfrac{1}{2}\operatorname{tr}\sigma$. A surface $\mathcal{M}$ is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. A surface is called *quasi-minimal* (or *pseudo-minimal*) if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H\rangle = 0$ [13].

3. Some Aspects on the General Rotational Surfaces

General rotational surfaces in the Euclidean 4-space $\mathbb{E}^4$ were introduced by F. N. Cole [14] and later studied by C. Moore [2] as follows. Let

$$\mu(s) = (x(s), y(s), z(s), w(s))\,; \quad s \in J \subset \mathbb{R} \tag{3.1}$$

be a smooth curve in $\mathbb{R}^4$, and a, b be real constants. A general rotational surface $\mathcal{M}$, generated by a rotation of the meridian curve $\mu(s)$, is immersed in $\mathbb{E}^4$ by

$$\mathcal{M} : Z(s,t) = \left(Z^1(s,t), Z^2(s,t), Z^3(s,t), Z^4(s,t)\right),$$

where

$$Z^1(s,t) = x(s)\cos at - y(s)\sin at; \qquad Z^3(s,t) = z(s)\cos bt - w(s)\sin bt;$$
$$Z^2(s,t) = x(s)\sin at + y(s)\cos at; \qquad Z^4(s,t) = z(s)\sin bt + w(s)\cos bt,$$

$t \in (0, 2\pi)$. The constants a, b determine the rates of rotation. Obviously, in the case $b = 0$, $y(s) = 0, s \in J$ one gets the classical rotation about a fixed two-dimensional axis. In [3], C. Moore described a special case of general rotational surfaces with constant Gauss curvature.

In [15], the second author studied general rotational surfaces whose meridians lie in two-dimensional planes and completely classified all minimal super-conformal general rotational surfaces. The minimal non-super-conformal general rotational surfaces with plane meridian curves were described by U. Dursun and N.C. Turgay in [16]. General rotational surfaces with plane meridian curves and pointwise 1-type Gauss map were studied in [4].

Similarly to the general rotations in $\mathbb{E}^4$, one can consider general rotational surfaces in the Minkowski 4-space $\mathbb{E}^4_1$ as follows. Let (3.1) be a smooth spacelike or timelike curve in $\mathbb{E}^4_1$, and a, b be real constants. Consider the surface immersed in $\mathbb{E}^4_1$ by

$$\mathcal{M} : Z(s,t) = \left(Z^1(s,t), Z^2(s,t), Z^3(s,t), Z^4(s,t)\right),$$

where

$$Z^1(s,t) = x(s)\cos at - y(s)\sin at; \qquad Z^3(s,t) = z(s)\cosh bt + w(s)\sinh bt;$$
$$Z^2(s,t) = x(s)\sin at + y(s)\cos at; \qquad Z^4(s,t) = z(s)\sinh bt + w(s)\cosh bt. \tag{3.2}$$

In the case $b = 0$ and either $y(s) = 0, s \in J$ or $x(s) = 0, s \in J$, one gets the standard rotational surface of elliptic type in $\mathbb{E}^4_1$. A local classification of spacelike rotational surfaces of elliptic type, whose mean curvature vector field is either vanishing or lightlike, was obtained in [17].

In the case $a = 0$, $z(s) = 0, s \in J$ we get the standard hyperbolic rotational surface of first type, and in the case $a = 0$, $w(s) = 0, s \in J$ we get the standard hyperbolic rotational surface of second type. A local classification of spacelike rotational surfaces of hyperbolic type with either vanishing or lightlike mean curvature vector field was given in [18]. In [19], the timelike and spacelike hyperbolic rotational surfaces with non-zero constant mean curvature in the three-dimensional de Sitter space $\mathbb{S}^3_1$ were classified. A classification of the spacelike and timelike Weingarten rotational surfaces in $\mathbb{S}^3_1$ was obtained in [20]. The class of Chen spacelike rotational surfaces of hyperbolic or elliptic type in $\mathbb{E}^4_1$ was described in [21].

In the case $a \neq 0$ and $b \neq 0$, the surfaces defined by (3.2) are analogous to the general rotational surfaces in the Euclidean space $\mathbb{E}^4$. In [8], G. Ganchev and the second author studied spacelike general rotational surfaces with plane meridian curves in $\mathbb{E}^4_1$ and described analytically the following main subclasses: minimal, flat, with flat normal connection. Spacelike general rotational surfaces in $\mathbb{E}^4_1$ with meridian curves lying in 2-dimensional planes and having pointwise 1-type Gauss map were studied in [22].

To go further with this subject studied in the Euclidean and the Minkowski case, it is natural for us to deal here with the general rotational surfaces in the pseudo-Euclidean 4-space with neutral metric $\mathbb{E}^4_2$. Let μ, defined by (3.1), be a spacelike (resp. timelike) curve in $\mathbb{E}^4_2$ parametrized by its arc-length s, which means that $\langle \dot{\mu}(s), \dot{\mu}(s)\rangle = \varepsilon$, i.e. $\dot{x}^2(s) + \dot{y}^2(s) - \dot{z}^2(s) - \dot{w}^2(s) = \varepsilon$, where $\varepsilon = 1$ in the spacelike case (resp. $\varepsilon = -1$ in the timelike case) and the dot denotes derivative with respect to s. We consider the surface $\mathcal{M}$ immersed in $\mathbb{E}^4_2$ as follows

$$\mathcal{M} : Z(s,t) = (Z_1(s,t), Z_2(s,t), Z_3(s,t), Z_4(s,t)), \tag{3.3}$$

where

$$\begin{aligned} Z_1(s,t) &= x(s)\cos at - y(s)\sin at; & Z_3(s,t) &= z(s)\cos bt - w(s)\sin bt;\\ Z_2(s,t) &= x(s)\sin at + y(s)\cos at; & Z_4(s,t) &= z(s)\sin bt + w(s)\cos bt, \end{aligned} \tag{3.4}$$

$s \in J$, $t \in (0, 2\pi)$, a, b are real constants, and $x(s), y(s), z(s), w(s)$ are smooth functions satisfying $\dot{x}^2(s) + \dot{y}^2(s) - \dot{z}^2(s) - \dot{w}^2(s) = \varepsilon$, where $\varepsilon = \pm 1$.

From now on, dot and prime will denote derivatives with respect to s and t, respectively, i.e. we will use the notations $\dot{Z}_i = \frac{\partial Z_i}{\partial s}$, $Z_i' = \frac{\partial Z_i}{\partial t}$.

Let us point out the following facts, obtained by direct calculation, which will be used in our further investigations.

Fact 1. The following conditions are equivalent:

(i) $\mathcal{M}$ is immersed in $\mathbb{E}^4_2$;

(ii) the matrix $\begin{pmatrix} \dot{Z}_1 & \dot{Z}_2 & \dot{Z}_3 & \dot{Z}_4 \\ Z_1' & Z_2' & Z_3' & Z_4' \end{pmatrix}$ is of rank 2;

(iii) at least one the the following determinants is non-zero:

$$\Delta_{ij} = \begin{vmatrix} \dot{Z}_i & \dot{Z}_j \\ Z_i' & Z_j' \end{vmatrix}, \quad i, j \in \{1, 2, 3\},\ i < j. \tag{3.5}$$

Fact 2. The determinants in (3.5) are given by:

$$\Delta_{12} = a(x\dot{x} + y\dot{y});$$

$$\begin{aligned}\Delta_{13} = &\cos at \cos bt(ay\dot{z} - b\dot{x}w) - \cos at \sin bt(b\dot{x}z + a\dot{w}y) + \\ &+ \sin at \cos bt(ax\dot{z} + b\dot{y}w) + \sin at \sin bt(b\dot{y}z - a\dot{w}x);\end{aligned}$$

$$\begin{aligned}\Delta_{14} = &\cos at \cos bt(ay\dot{w} + b\dot{x}z) - \cos at \sin bt(ay\dot{z} - b\dot{x}w) + \\ &+ \sin at \cos bt(ax\dot{w} - b\dot{y}z) + \sin at \sin bt(ax\dot{z} + b\dot{y}w);\end{aligned}$$

$$\begin{aligned}\Delta_{23} = &-\cos at \cos bt(ax\dot{z} + b\dot{y}w) + \cos at \sin bt(ax\dot{w} - b\dot{y}z) - \\ &- \sin at \cos bt(ay\dot{z} - b\dot{x}w) - \sin at \sin bt(ay\dot{w} + b\dot{x}z);\end{aligned}$$

$$\begin{aligned}\Delta_{24} = &-\cos at \cos bt(ax\dot{w} - b\dot{y}z) - \cos at \sin bt(ax\dot{z} + b\dot{y}w) + \\ &+ \sin at \cos bt(ay\dot{w} + b\dot{x}z) + \sin at \sin bt(ay\dot{z} - b\dot{x}w);\end{aligned}$$

$$\Delta_{34} = b(z\dot{z} + w\dot{w}).$$

By using some product-to-sum trigonometric identities we obtain the following

Fact 3. Let a and b be two non-zero real constants and A, B, C, and D be some functions independent of $t \in (0; 2\pi)$.

(i) If $a \neq b$, then

$$A \cos at \cos bt + B \cos at \sin bt + C \sin at \cos bt + D \sin at \sin bt = 0, \ \ t \in (0; 2\pi)$$

if and only if $A = B = C = D = 0$;

(ii) If $a = b$, then

$$A \cos^2 at + (B + C) \sin at \cos at + D \sin^2 at = 0, \ \ t \in (0; 2\pi)$$

if and only if $A = B + C = D = 0$.

The three facts above underlie the following two statements.

Proposition 3.1. *The following conditions are equivalent:*

(i) $\Delta_{ij} = 0$ *for all* $i, j = 1, 2, 3$ *and* $i < j$*;*

(ii) $x\dot{x} + y\dot{y} = z\dot{z} + w\dot{w} = ay\dot{z} - b\dot{x}w = ay\dot{w} + b\dot{x}z = ax\dot{z} + b\dot{y}w = ax\dot{w} - b\dot{y}z = 0$*;*

(iii) The functions x*,* y*,* z*,* w *in the parametrization* (3.3) *and* (3.4) *of* $Z(s, t)$ *are either constant or determined by*

$$x = c \cos\left(\frac{a}{b}u(s) + q\right), \quad y = c \sin\left(\frac{a}{b}u(s) + q\right), \tag{3.6}$$

$$z = k \cos u(s), \quad w = k \sin u(s), \tag{3.7}$$

where a*,* b *are non-zero real constants,* c*,* k*,* $q \in \mathbb{R}$ *and* $u(s)$ *is a smooth function.*

Proposition 3.2. *If* $a = 0$ *and* b *is a non-zero constant, then the following conditions are equivalent:*

(i) $\Delta_{ij} = 0$*, for all* $i = 1, 2, 3$ *and* $i < j$*;*

(ii) $z\dot{z} + w\dot{w} = w\dot{x} = z\dot{x} = w\dot{y} = z\dot{y} = 0$*;*

(iii) The functions x, y, z, w in the parametrization (3.3) *and* (3.4) *of* $Z(s,t)$ *belong to one of the following cases:*
(a): $x(s)$ *is non-constant,* $y(s)$ *is an arbitrary smooth function,* $z = w = 0$*;*
(b): $x(s)$ *is an arbitrary smooth function,* $y(s)$ *is non-constant,* $z = w = 0$*;*
(c): $x = c$, $y = q$, $z = k\cos u$, $w = k\sin u$, *where* $c, k, q \in \mathbb{R}$ *and* $u(s)$ *is a smooth function.*

Remark 3.3. (1) If a is a non-zero constant and $b = 0$, then the statement of Proposition 3.2 remains valid if we replace x, y, z, w respectively by z, w, x, y.
(2) If $a = b = 0$, then $\Delta_{ij} = 0$, for all $i = 1, 2, 3$ and $i < j$.

To summarize, Proposition 3.1, Proposition 3.2, and Remark 3.3 give the following result.

Proposition 3.4. *Let $\mathcal{M}$ be a general rotational surface given by* (3.3) *and* (3.4)*. Then $\mathcal{M}$ is immersed in $\mathbb{E}^4_2$ if and only if none of the following conditions holds:*

(i) $a = b = 0$*;*

(ii) $a \neq 0$, $b \neq 0$, x, y, z, w *are constants;*

(iii) $a \neq 0$, $b \neq 0$, $x = c\cos\left(\frac{a}{b}u(s) + q\right)$, $y = c\sin\left(\frac{a}{b}u(s) + q\right)$, $z = k\cos u(s)$, $w = k\sin u(s)$*;*

(iv) $a = 0$, $b \neq 0$, $z = w = 0$, *and either x or y is non-constant;*

(v) $a = 0$, $b \neq 0$, $x = c$, $y = q$, $z = k\cos u$, $w = k\sin u$, *for* $c, k, q \in \mathbb{R}$ *and some smooth function* $u(s)$*;*

(vi) $a \neq 0$, $b = 0$, $x = y = 0$, *and either z or w is non-constant;*

(vii) $a \neq 0$, $b = 0$, $z = c$, $w = q$, $x = k\cos u$, $y = k\sin u$, *for* $c, k, q \in \mathbb{R}$ *and some smooth function* $u(s)$.

Further, we will assume that $\mathcal{M}$ is immersed in $\mathbb{E}^4_2$.

Some particular cases of a general rotational surfaces are of special interest:

I. In the case $b = 0$, and either $y(s) = 0$ or $x(s) = 0$ (resp. $a = 0$, and either $z(s) = 0$ or $w(s) = 0$) the surface $\mathcal{M}$ parametrized by (3.3) and (3.4) becomes a standard rotational surface of elliptic type in $\mathbb{E}^4_2$.

II. In the case $a \neq 0$, $b \neq 0$, $y(s) = 0$, $w(s) = 0$, the meridian curve $\mu(s)$ is a plane curve (lying in the two-dimensional xz-plane) and the general rotational surface $\mathcal{M}$ has the following parametrization:

$$\mathcal{M} : Z(s,t) = (x(s)\cos at, x(s)\sin at, z(s)\cos bt, z(s)\sin bt)\,. \tag{3.8}$$

Lorentz general rotational surfaces in $\mathbb{E}^4_2$ with plane meridian curves are studied in [12] where the complete classification of some special subclasses is given, namely: minimal, flat, with flat normal connection, with parallel normalized mean curvature vector field. A particular subcase is the analogue of the Vranceanu rotational surface in the Euclidean space $\mathbb{E}^4$ which is obtained choosing $a = b = 1$, $x(s) = u(s)\cos s$, and $z(s) = u(s)\sin s$ for some smooth function $u(s)$ [23].

Remark 3.5. If the surface $\mathcal{M}$ is parametrized by (3.3) and (3.4) and the meridian curve $\mu(s)$ satisfies $\frac{x(s)}{y(s)} = const$ and $\frac{z(s)}{w(s)} = const$, then we can find functions $f(s)$ and $g(s)$ and re-parametrize the surface in the following form:

$$(f(s)\cos(at + a_0), f(s)\sin(at + a_0), g(s)\cos(bt + b_0), g(s)\sin(bt + b_0)),$$

which is equivalent to (3.8) up to constants a_0 and b_0. In the present paper, we study the general case, i.e. $\frac{x(s)}{y(s)} \neq const$ or $\frac{z(s)}{w(s)} \neq const$.

The general rotational surfaces in $\mathbb{E}^4_2$ parametrized by (3.3) and (3.4) are of elliptic type (the elliptic cosine and sine functions are used in the parametrization). Similar to the general rotational surfaces of elliptic type, in [12], Aleksieva, Turgay and the second author define general rotational surfaces of hyperbolic type in $\mathbb{E}^4_2$ as follows:

$$\begin{aligned} Z_1(s,t) &= x(s)\cosh at + z(s)\sinh at; & Z_3(s,t) &= x(s)\sinh at + z(s)\cosh at; \\ Z_2(s,t) &= y(s)\cosh bt + w(s)\sinh bt; & Z_4(s,t) &= y(s)\sinh bt + w(s)\cosh bt. \end{aligned} \tag{3.9}$$

In the case $b = 0$ and $z(s) = 0$, one obtains the surface with parametrization

$$Z(s,t) = (x(s)\cosh at, y(s), x(s)\sinh at, w(s)),$$

which is a standard rotation of hyperbolic type about the yw-axis. In the case $a = 0$ and $w(s) = 0$, one gets the surface

$$Z(s,t)) = (x(s), y(s)\cosh bt, z(s), y(s)\sinh bt),$$

which is a standard rotation of hyperbolic type about the xz-axis. If both a and b are non-zero constants the surface defined by (3.9) is a general rotational surface of hyperbolic type in $\mathbb{E}^4_2$. Lorentz general rotational surfaces of hyperbolic type with plane meridian curves are studied in [12].

General rotational surfaces with plane meridian curves and pointwise 1-type Gauss map in $\mathbb{E}^4_2$ are studied in [11] in the case the Gauss curvature is zero and $a = b = 1$. In [24], the classification of general rotational surfaces having zero mean curvature and pointwise 1-type Gauss map of second kind is given.

In all mentioned papers [11,12,24], only the case of meridian surfaces lying in two-dimensional planes is considered. In what follows, we study general rotational surfaces of elliptic type in $\mathbb{E}^4_2$ whose meridian curves lie in 4-dimensional space. Similar considerations can be made for general rotational surfaces of hyperbolic type (in the general case, when the meridian curves do not lie in two-dimensional planes), but we are not including them in the present paper for conciseness.

Let $\mathcal{M}$ be a general rotational surface immersed in $\mathbb{E}^4_2$ and parametrized by (3.3) and (3.4). Then, the tangent frame field of $\mathcal{M}$ is determined by the following vector fields:

$$Z_s = (\dot{x}\cos at - \dot{y}\sin at, \dot{x}\sin at + \dot{y}\cos at, \dot{z}\cos bt - \dot{w}\sin bt, \dot{z}\sin bt + \dot{w}\cos bt);$$

$$Z_t = (-ax\sin at - ay\cos at, ax\cos at - ay\sin at, -bz\sin bt - bw\cos bt, bz\cos bt - bw\sin bt).$$

The coefficients of the induced metric on $\mathcal{M}$ are expressed by:

$$E = \langle Z_s, Z_s\rangle = \varepsilon = \pm 1;$$

$$F = \langle Z_s, Z_t\rangle = a(x\dot{y} - y\dot{x}) + b(w\dot{z} - z\dot{w});$$

$$G = \langle Z_t, Z_t\rangle = a^2(x^2 + y^2) - b^2(z^2 + w^2).$$

If we denote $\Delta = EG - F^2$, then the function Δ is expressed by

$$\begin{aligned}\Delta = a^2(x\dot{x} + y\dot{y})^2 + b^2(z\dot{z} + w\dot{w})^2 - (ax\dot{z} + bw\dot{y})^2 \\ - (ay\dot{z} - bw\dot{x})^2 - (ax\dot{w} - bz\dot{y})^2 - (ay\dot{w} + bz\dot{x})^2.\end{aligned}$$

Definition 3.6. Any semi-Riemannian metric on a manifold induces on a submanifold a metric of type (p, q, r), where p, q, r represent the positive, the negative and the zero eigenvalues, respectively. The submanifold itself and this metric restricted to the submanifold are called:

- *non-degenerate*, if its nullity degree vanishes, i.e. $r = 0$;
- *spacelike*, if it is non-degenerate of index $q = 0$;
- *timelike*, if it is non-degenerate of index $q \geq 1$, and in particular, when $q = 1$, it is called *Lorentz*;
- *lightlike*, if it is degenerate, i.e. $r > 0$.

Remark 3.7. Based on the above definition, a surface of a semi-Riemannian manifold is lightlike if and only if its metric, restricted to each point of the surface, is either identically zero, or is of rank one, i.e. $p + q = 1$.

Using the terminology from [25] and the above expressions of E, F, G, Δ, we obtain the following statements, which characterize the character of $\mathcal{M}$.

Proposition 3.8. *Let $\mathcal{M}$ be a general rotational surface given by* (3.3) *and* (3.4)*. Then, we have the following equivalences:*

(a) $\mathcal{M}$ is spacelike $\Longleftrightarrow$ μ is spacelike and $\Delta > 0 \Longleftrightarrow \varepsilon = 1$ and $\Delta > 0$;

(b) $\mathcal{M}$ is of index 2 $\Longleftrightarrow$ μ is timelike and $\Delta < 0 \Longleftrightarrow \varepsilon = -1$ and $\Delta < 0$;

(c) $\mathcal{M}$ is Lorentz $\Longleftrightarrow$ either μ is spacelike and $\Delta < 0$ or μ is timelike and $\Delta > 0 \Longleftrightarrow \varepsilon\Delta < 0$.

Proposition 3.9. *Let $\mathcal{M}$ be a general rotational surface given by* (3.3) *and* (3.4)*. Then, $\mathcal{M}$ is lightlike (i.e. degenerate) if and only if $\Delta = 0$.*

Remark 3.10. We emphasize that the restriction of the pseudo-Euclidean metric of $\mathbb{E}^4_2$ to the general rotational surface $\mathcal{M}$ is never identically zero, since we assumed μ, given by (3.1), is either spacelike or timelike. Hence, on general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$, Remark 3.7 applies only when the restricted metric is of rank 1.

4. General Rotational Surfaces with Orthogonal Parametric Lines

This section is devoted to the study of general rotational surfaces whose parametric lines are orthogonal.

Proposition 4.1. *The tangent vector fields Z_s and Z_t of a general rotational surface $\mathcal{M}$ given by* (3.3) *and* (3.4) *are orthogonal if and only if one of the following cases occurs:*

(i) $b = 0$ *and either* $x = 0$ *or* $y = 0$*;*

(ii) $b = 0$ *and* $y = cx$, $c \neq 0$*;*

(iii) $b \neq 0$, $z = 0$ *and either* $x = 0$ *or* $y = 0$*;*

(iv) $b \neq 0$, $z = 0$ *and* $y = cx$*;*

(v) $b \neq 0$ *and* $w(s) = (c + \varphi(s))\, z(s)$ *for some constant* $c \in \mathbb{R}$ *and*

$$\varphi(s) = \frac{a}{b} \int \frac{x\dot{y} - y\dot{x}}{z^2} ds. \tag{4.1}$$

Proof. We will analyse the following possible cases:

Case 1. The constant b is zero. Then, $F = 0$ if and and only if $x\dot{y} = y\dot{x}$ and hence, either we have $x = 0$ or $y = 0$, i.e. *(i)* occurs, or x and y are both non-zero functions and $y = cx$ for some constant c, i.e. *(ii)* occurs.

Case 2. The constant b is non-zero. Then, denoting $u = \frac{a}{b}\, y$ we obtain that $F = 0$ if and only if the following differential equation holds:

$$z\dot{w} - w\dot{z} = x\dot{u} - u\dot{x}. \tag{4.2}$$

Now, we consider (4.2) as a differential equation of order 1 with respect to w. If we assume that $z = 0$, then $x\dot{u} = u\dot{x}$, or equivalently $x\dot{y} = y\dot{x}$, which implies that either *(iii)* or *(iv)* occurs. If $z \neq 0$, then the general solution of (4.2) is given by $w = w_0 + w_p$, where $w_0 = cz$, $c = const$, is the general solution of the homogeneous equation $z\dot{w} - w\dot{z} = 0$ and $w_p = \varphi(s)z$ is a particular solution of (4.2), where the function $\varphi(s)$ is given by (4.1).

□

A special case of orthogonal parametric lines is the case of isothermal parametrization. A surface is said to be parametrized by isothermal parameters, if $F = 0$, and $E = G$ in case of a spacelike surface (resp. $E = -G$ in case of a timelike surface), i.e. the tangent vector fields Z_s and Z_t are orthogonal and have equal length (up to a sign). As a consequence of Proposition 4.1 we obtain the following result.

Proposition 4.2. *The general rotational surface $\mathcal{M}$ given by* (3.3) *and* (3.4) *is parametrized by isothermal coordinates* (s, t) *if and only if one of the following conditions is satisfied:*

(i) $b = 0, \quad x = \pm\frac{1}{a\sqrt{1+c^2}}, \quad y = cx$*;*

(ii) $b = 0, \quad x = 0, \quad y = \pm\frac{1}{a}$*;*

(iii) $b = 0, \quad x = \pm\frac{1}{a}, \quad y = 0$;
(iv) $b \neq 0, \quad a^2(x^2 + y^2) - b^2 z^2 \left(1 + (c + \varphi)^2\right) = \pm 1, \quad w = (c + \varphi)\, z$;
(v) $b \neq 0, \quad x = 0, \quad z = 0, \quad w = \pm\frac{\sqrt{a^2 y^2 \mp 1}}{b}$;
(vi) $b \neq 0, \quad y = 0, \quad z = 0, \quad w = \pm\frac{\sqrt{a^2 x^2 \mp 1}}{b}$
for some $c \in \mathbb{R}$ and a smooth function $\varphi(s)$ given by (4.1).

To study in more details the general rotational surfaces whose geometric parametric lines are orthogonal, we will consider separately each case mentioned in Proposition 4.1.

Case *(i)*: If $b = 0$ and $y = 0$, then the surface $\mathcal{M}$ is parametrized by

$$\mathcal{M} : Z(s,t) = (x(s)\cos at, x(s)\sin at, z(s), w(s))\,,$$

which is the parametrization of a standard rotational surface of elliptic type. Similarly for the case $b = 0$ and $x = 0$.

Case *(ii)*: If $b = 0$ and $y = cx$, $c \neq 0$, then the surfaces has the following parametrization:

$$\mathcal{M} : Z(s,t) = (x(s)(\cos at - c\sin at), x(s)(\sin at + c\cos at), z(s), w(s))\,. \quad (4.3)$$

We denote

$$g_1(t) = \cos at - c\sin at, \quad g_2(t) = \sin at + c\cos at.$$

Then, $g_1^2 + g_2^2 = 1 + c^2$, and hence, there exists a function $\psi(t)$ such that

$$g_1(t) = \sqrt{1 + c^2}\cos\psi(t)$$
$$g_2(t) = \sqrt{1 + c^2}\sin\psi(t)$$

Indeed, the above equalities imply that $\tan\psi(t) = \frac{\tan at + c}{1 - c\tan at}$. So, if we consider the function $\psi(t)$, defined by

$$\psi(t) = \arctan\frac{\tan at + c}{1 - c\tan at},$$

and denote $r(s) = \sqrt{1 + c^2}\, x(s)$, then the parametrization (4.3) of $\mathcal{M}$ takes the form:

$$\mathcal{M} : Z(s,\psi) = (r(s)\cos\psi, r(s)\sin\psi, z(s), w(s))\,,$$

which shows that $\mathcal{M}$ is a standard rotational surface of elliptic type in $\mathbb{E}^4_2$.

Case *(iii)*: If $b \neq 0$, $z = 0$ and $y = 0$, then the surface $\mathcal{M}$ is parametrized by

$$\mathcal{M} : Z(s,t) = (x(s)\cos at, x(s)\sin at, -w(s)\sin bt, w(s)\cos wt)\,,$$

which is the parametrization of a general rotational surface of elliptic type with plane meridian curves. Similarly for the case $b \neq 0$, $z = 0$ and $x = 0$.

Case *(iv)*: If $b \neq 0$, $z = 0$ and $y = cx$, $c \neq 0$, then the surface has the following parametrization:

$$\mathcal{M} : Z(s,t) = \Big(x(s)(\cos at - c\sin at), x(s)(\sin at + c\cos at), \\ - w(s)\sin bt, w(s)\cos bt \Big).$$

Denoting

$$g_1(t) = \cos at - c\sin at, \quad g_2(t) = \sin at + c\cos at,$$

we get $g_1^2 + g_2^2 = 1 + c^2$. Hence, we can consider the function $\psi(t)$, defined by

$$\psi(t) = \arctan \frac{\tan at + c}{1 - c\tan at},$$

and denote $r(s) = \sqrt{1+c^2}\, x(s)$. Then, we obtain the following parametrization of $\mathcal{M}$:

$$\mathcal{M} : Z(s,t) = \left(r(s)\cos\psi(t), r(s)\sin\psi(t), -w(s)\sin bt, w(s)\cos bt \right). \tag{4.4}$$

Setting $\tilde{c} = \arctan c$, we get $\tan\psi(t) = \tan(at + \tilde{c})$ and hence, $\psi(t) = at + \tilde{c} + k\pi$, $k \in \mathbb{Z}$. If we assume that $a \in \mathbb{Z}$ and $\tilde{c} = -ak\pi$ for some $k \in \mathbb{Z}$, then we get $c = 0$, which contradicts the condition $c \neq 0$. If $a \notin \mathbb{Z}$ and $-\frac{\arctan c}{a\pi} \in \mathbb{Z}$, then the parametrization (4.4) takes the form

$$\mathcal{M} : Z(s,\psi) = \left(r(s)\cos\psi, r(s)\sin\psi, -w(s)\sin\tilde{a}\psi, w(s)\cos\tilde{a}\psi \right),$$

where $\tilde{a} = \frac{b}{a}$. Consequently, $\mathcal{M}$ is a general rotational surface of elliptic type with plane meridian curves.

Case *(v)*: If $b \neq 0$ and $w(s) = (c + \varphi(s))\, z(s)$, $c \neq 0$, where

$$\varphi(s) = \frac{a}{b} \int \frac{x\dot{y} - y\dot{x}}{z^2} ds,$$

then denoting $\tilde{\varphi}(s) = c + \varphi(s)$, we get

$$z(s)\cos bt - w(s)\sin bt = z(s)(\cos bt - \tilde{\varphi}(s)\sin bt);$$

$$z(s)\sin bt + w(s)\cos bt = z(s)(\sin bt + \tilde{\varphi}(s)\cos bt).$$

So, in this case we have $(Z_3(s,t))^2 + (Z_4(s,t))^2 = z^2(s)(1 + \tilde{\varphi}^2(s))$, which is not constant in general.

But obviously, $\tilde{\varphi}(s) = const \iff \varphi(s) = const \iff \frac{y(s)}{x(s)} = const$. We consider the subcase $y(s) = c_1 x(s)$, $c_1 \neq 0$. Then $\varphi = 0$ and $\tilde{\varphi} = c$. For convenience, we denote $\tilde{\varphi}$ by c_2, and hence $w(s) = c_2 z(s)$, $c_2 \neq 0$. In this subcase, the surface $\mathcal{M}$ is parametrized by:

$$Z(s,t) = \Big(x(\cos at - c_1\sin at), x(\sin at + c_1\cos at), \\ z(\cos bt - c_2\sin bt), z(\sin bt + c_2\cos bt) \Big).$$

Now, denoting

$$g_1(t) = \cos at - c_1 \sin at; \qquad g_3(t) = \cos bt - c_2 \sin bt;$$
$$g_2(t) = \sin at + c_1 \cos at; \qquad g_4(t) = \sin bt + c_2 \cos bt,$$

we obtain that the parametrization of the surface can be rewritten as follows:

$$Z(s,t) = (x(s)g_1(t), x(s)g_2(t), z(s)g_3(t), z(s)g_4(t)).$$

Moreover, we have $g_1^2 + g_2^2 = 1 + c_1^2$, $g_3^2 + g_4^2 = 1 + c_2^2$. Hence, we can consider the functions $\psi_1(t)$ and $\psi_2(t)$, defined by

$$\psi_1(t) = \arctan \frac{\tan at + c_1}{1 - c_1 \tan at}; \qquad \psi_2(t) = \arctan \frac{\tan bt + c_2}{1 - c_2 \tan bt}.$$

Then, the functions $g_i(t)$, $i = 1, 2, 3, 4$ can be written as

$$g_1(t) = \sqrt{1 + c_1^2} \cos \psi_1(t); \qquad g_3(t) = \sqrt{1 + c_2^2} \cos \psi_2(t);$$
$$g_2(t) = \sqrt{1 + c_1^2} \sin \psi_1(t); \qquad g_4(t) = \sqrt{1 + c_2^2} \sin \psi_2(t).$$

Setting $\tilde{c}_1 = \arctan c_1$, $\tilde{c}_2 = \arctan c_2$, we get

$$\tan \psi_1(t) = \tan(at + \tilde{c}_1), \qquad \tan \psi_2(t) = \tan(bt + \tilde{c}_2),$$

which imply that $\psi_1(t) = at + \tilde{c}_1 + k_1\pi$, $\psi_2(t) = bt + \tilde{c}_2 + k_2\pi$, where $k_1, k_2 \in \mathbb{Z}$.

Obviously, $\frac{\psi_1}{\psi_2} = const \iff \frac{b}{a} = \frac{\arctan c_2 + k_2\pi}{\arctan c_1 + k_1\pi}$ for some $k_1, k_2 \in \mathbb{Z}$. In the special subcase in which the constants c_1 and c_2 satisfy the relation

$$\frac{b}{a} = \frac{\arctan c_2 + k_2\pi}{\arctan c_1 + k_1\pi}$$

for some $k_1, k_2 \in \mathbb{Z}$, we may consider the following change of the parameter t:

$$\psi_2(t) = \arctan \frac{\tan bt + c_2}{1 - c_2 \tan bt}.$$

Now, we set $\psi_1 = \tilde{a}\psi_2$, where $\tilde{a} = \frac{at + \tilde{c}_1 + k_1\pi}{bt + \tilde{c}_2 + k_2\pi} = const$. Denote

$$\tilde{x}(s) = \sqrt{1 + c_1^2}\, x(s), \qquad \tilde{z}(s) = \sqrt{1 + c_2^2}\, z(s).$$

Then, we obtain the following parametrization of the surface:

$$\mathcal{M} : Z(s, \psi_2) = (\tilde{x}(s) \cos \tilde{a}\psi_2, \tilde{x}(s) \sin \tilde{a}\psi_2, \tilde{z}(s) \cos \psi_2, \tilde{z}(s) \sin \psi_2).$$

Consequently, $\mathcal{M}$ is a general rotational surface of elliptic type with plane meridian curves.

Finally, we reach to the following conclusions:

1. Cases *(i)* and *(ii)* correspond to the class of standard rotational surfaces (up to re-parametrization).

2 . In cases *(iii)*, *(iv)*, and *(v)*, there exists a subclass which corresponds to the class of general rotational surfaces of elliptic type with plane meridian curves (up to re-parametrization).

5. Killing Vector Fields on General Rotational Surfaces

Let us recall that on a semi-Riemannian manifold (N, g), a vector ξ is called *Killing*, provided the Lie derivative $\mathcal{L}_\xi g$ vanishes identically. In this section, we will study Killing vector fields on general rotational surfaces in $\mathbb{E}^4_2$.

On a general rotational surface $\mathcal{M}$ given by (3.3) and (3.4), the second derivatives of $Z(s,t)$ are given by:

$$Z_{ss} = (\ddot{x}\cos at - \ddot{y}\sin at, \ddot{x}\sin at + \ddot{y}\cos at, \ddot{z}\cos bt - \ddot{w}\sin bt, \ddot{z}\sin bt + \ddot{w}\cos bt);$$

$$Z_{st} = (-a\dot{x}\sin at - a\dot{y}\cos at, a\dot{x}\cos at - a\dot{y}\sin at, -b\dot{z}\sin bt - b\dot{w}\cos bt, b\dot{z}\cos bt - b\dot{w}\sin bt);$$

$$Z_{tt} = (-a^2x\cos at + a^2y\sin at, -a^2x\sin at - a^2y\cos at, -b^2z\cos bt + b^2w\sin bt, -b^2z\sin bt - b^2w\cos bt).$$

If ∇ denotes the Levi-Civita connection on the surface $\mathcal{M}$, then

$$\begin{array}{l} \langle \nabla_{Z_s} Z_s, Z_s\rangle = \langle \nabla_{Z_s} Z_t, Z_s\rangle = \langle \nabla_{Z_t} Z_t, Z_t\rangle = 0; \\ \langle \nabla_{Z_s} Z_s, Z_t\rangle = \dot{F}; \\ \langle \nabla_{Z_s} Z_t, Z_t\rangle = -\langle \nabla_{Z_t} Z_t, Z_s\rangle = \frac{\dot{G}}{2}, \end{array} \tag{5.1}$$

and the functions $\dot{F}$, $\dot{G}$ are expressed in terms of x, y, z, w as follows:

$$\dot{F} = a(x\ddot{y} - y\ddot{x}) - b(z\ddot{w} - w\ddot{z}), \tag{5.2}$$

$$\dot{G} = 2\left(a^2(x\dot{x} + y\dot{y}) - b^2(z\dot{z} + w\dot{w})\right). \tag{5.3}$$

Further, we denote $f = \dot{F}$ and $g = \frac{\dot{G}}{2}$.

Let $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ be a vector field tangent to the general rotational surface $\mathcal{M}$, α and β being arbitrary smooth functions of (s,t). For any X and Y tangent to $\mathcal{M}$ the Lie derivative $\mathcal{L}_\xi$ with respect to ξ is given by

$$\left(\mathcal{L}_\xi\langle,\rangle\right)(X,Y) = \langle \nabla_\xi X, Y\rangle + \langle X, \nabla_\xi Y\rangle - \langle [\xi, X], Y\rangle - \langle X, [\xi, Y]\rangle.$$

Then, from $[\xi, Z_s] = -\dot{\alpha}Z_s - \dot{\beta}Z_t$, $[\xi, Z_t] = -\alpha' Z_s - \beta' Z_t$, and (5.1), we obtain

$$\begin{array}{l} \left(\mathcal{L}_\xi\langle,\rangle\right)(Z_s, Z_s) = 2(\varepsilon\dot{\alpha} + F\dot{\beta}); \\ \left(\mathcal{L}_\xi\langle,\rangle\right)(Z_s, Z_t) = \alpha f + \varepsilon\alpha' + (\dot{\alpha} + \beta')F + G\dot{\beta}; \\ \left(\mathcal{L}_\xi\langle,\rangle\right)(Z_t, Z_t) = \alpha\dot{G} + 2(\alpha' F + \beta' G). \end{array} \tag{5.4}$$

In the following statement, we consider vector fields with separable variables, simplifying analysis in fields such as fluid dynamics or electromagnetism. This separability allows complex equations to be decomposed into simpler, independent equations for each variable, making them solvable using separation of variables techniques, which often lead to exact solutions. We prefer to work in the context of separable variables in order to make the exposition clearer.

Theorem 5.1. *Any Killing vector field $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ tangent to the general rotational surface $\mathcal{M}$, for which $\alpha(s,t)$ has separable variables, falls into one of the following four cases:*

(i) $\mathcal{M}$ *is lightlike,* $\alpha = ce^{-\varepsilon f t}$ *for some constant* $c \in \mathbb{R}$*, and* $\beta(s,t)$ *is an arbitrary smooth function;*

(ii) $\mathcal{M}$ *is semi-Riemannian (non-degenerate),* $\alpha = ce^{A(s)}$*,* $\beta = -\varepsilon c \int \frac{\dot{A}e^{A(s)}}{F} ds$ *for some constant* $c \in \mathbb{R}$ *and* $A(s) = \int \frac{fF}{\Delta} ds$*;*

(iii) $\mathcal{M}$ *is lightlike,* $f = 0$*,* $\alpha = ce^{A(s)}$*,* $\beta = -\varepsilon c \int \frac{\dot{A}e^{A(s)}}{F} ds$ *for some constant* $c \in \mathbb{R}$ *and an arbitrary function* $A(s)$*;*

(iv) $\mathcal{M}$ *is semi-Riemannian (non-degenerate),* $\alpha = cT'(t)$*,* $\beta = -\frac{c}{2}\left((\dot{\ln G})T(t) + U(s)\right)$ *for some constant* $c \in \mathbb{R}$ *and*

$$T(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + g_1(t)e^{\lambda_1 t} + g_2(t)e^{\lambda_2 t}, \tag{5.5}$$

where $c_1, c_2 \in \mathbb{R}$*,* $\lambda_{1,2}(s) = \varepsilon \frac{-f \pm \sqrt{f^2 + 2\varepsilon G (\ddot{\ln G})}}{2}$*, and* $g_1(t), g_2(t)$ *are particular solutions such that (5.5) satisfies the following second order differential equation with respect to* t *with constant coefficients:*

$$\varepsilon T'' + fT' - \frac{G}{2}(\ddot{\ln G})T = U(s)G. \tag{5.6}$$

Remark 5.2. Based on Remark 3.10, it makes sense to consider Killing vector fields when $\mathcal{M}$ is lightlike, since the pseudo-Euclidean metric restricted to $\mathcal{M}$ is never identically zero.

Proof. It follows from (5.4) that ξ is a Killing vector field if and only if the following system holds:

$$\begin{aligned} &\varepsilon\dot{\alpha} + F\dot{\beta} = 0 \\ &f\alpha + \varepsilon\alpha' + F\dot{\alpha} + F\beta' + G\dot{\beta} = 0 \\ &\dot{G}\alpha + 2F\alpha' + 2G\beta' = 0 \end{aligned} \tag{5.7}$$

Having separable variables, $\alpha(s,t)$ can be written as $\alpha(s,t) = e^{A(s)+B(t)}$ for some smooth functions A and B.

Case I. If $F = G = 0$, then $\Delta = 0$ and from Proposition 3.9 it follows that $\mathcal{M}$ is lightlike. In this case, system (5.7) takes the form

$$\begin{aligned} &\varepsilon\dot{\alpha} = 0 \\ &f\alpha + \varepsilon\alpha' = 0 \end{aligned}$$

which is equivalent to *(i)*.

Case II. If $F \neq 0$ and $G \neq 0$, then replacing $\dot{\beta}$ and β' from the first and respectively third equation of (5.7) in the second one, we obtain

$$\alpha\left(f - \frac{\dot{G}F}{2G}\right) + \alpha'\frac{\Delta}{G} - \dot{\alpha}\frac{\Delta}{F} = 0. \tag{5.8}$$

Using that $\alpha(s,t) = e^{A(s)+B(t)}$, we get

$$f - \frac{\dot{G}F}{2G} + B'\frac{\Delta}{G} - \dot{A}\frac{\Delta}{F} = 0,$$

which implies $B' = 0$ and hence, $\alpha = ce^{A(s)}$ for some constant $c \in \mathbb{R}$. Then, the first equation of system (5.7) implies that

$$\beta = -\varepsilon c \int \frac{\dot{A}e^{A}}{F} ds + T(t)$$

for some smooth function $T(t)$. From the last equation of system (5.7) it follows that T is constant and hence, we may write β in the following form $\beta = -\varepsilon c \int \frac{\dot{A}e^{A(s)}}{F} ds$.

The third equation of system (5.7) takes the form $\alpha\dot{G} = 0$, which implies that either $c = 0$ or $G = q$ for a non-zero constant q.

a) If $c = 0$, then the Killing vector field ξ is trivial.

b) If $c \neq 0$, then $G = q$ and the system reduces to

$$\begin{aligned} &\alpha = ce^{A(S)} \\ &\beta = -\varepsilon c \textstyle\int \frac{\dot{A}e^{A}}{F} ds \\ &\dot{A}\Delta = fF \end{aligned} \tag{5.9}$$

which implies that either $\Delta = 0$ (and hence $f = 0$) and therefore *(iii)* occurs, or $\Delta \neq 0$ and therefore *(ii)* occurs.

Case III. If $F = 0$ and $G \neq 0$, then system (5.7) reduces to

$$\begin{aligned} &\varepsilon\dot{\alpha} = 0 \\ &f\alpha + \varepsilon\alpha' + G\dot{\beta} = 0 \\ &\dot{G}\alpha + 2G\beta' = 0 \end{aligned}$$

or equivalently

$$\begin{aligned} &\alpha = ce^{B(s)} \\ &ce^{B}(f + \varepsilon B') + G\dot{\beta} = 0 \\ &\beta = -\tfrac{c}{2}\left(\tfrac{\dot{G}}{G}\textstyle\int e^{B}dt + U(s)\right) \end{aligned}$$

for some constant $c \in \mathbb{R}$ and some smooth function $U(s)$. Denoting $T(t) = \int e^{B(t)}dt$ we obtain *(iv)*.

Case IV. If $F \neq 0$ and $G = 0$, then system (5.7) reduces to

$$\begin{aligned} &\varepsilon\dot{\alpha} + F\dot{\beta} = 0 \\ &f\alpha + \varepsilon\alpha' + F\dot{\alpha} + F\beta' = 0 \\ &F\alpha' = 0 \end{aligned}$$

which is equivalent to

$$\alpha = ce^{A(s)}$$
$$\beta = -c\varepsilon \int \frac{\dot{A}e^{A(s)}}{F} ds + k(t)$$
$$ce^{A} f + c\dot{A}e^{A} F + k' F = 0$$

for a smooth function $k(t)$. From the last equation it follows that $k(t)$ is a constant. Hence, we obtain *(ii)* and complete the proof.

□

Remark 5.3. We may obtain a similar statement if in Theorem 5.1, instead of $\alpha(s,t)$, we assume that $\beta(s,t)$ has separable variables.

6. Killing Vector Fields on Standard Rotational Surfaces

The present section is devoted to the standard rotational surfaces as a special case of general rotational surfaces, mainly with the purpose of giving more information about Killing vector fields by providing detailed calculations for integration.

A standard rotational surface $\mathcal{M}$ of elliptic type in $\mathbb{E}^4_2$ is defined by

$$\mathcal{M} : Z(s,t) = (x(s), y(s), r(s)\cos t, r(s)\sin t), \quad t \in (0,\pi), \tag{6.1}$$

for some smooth functions $x(s)$, $y(s)$, $r(s)$. Without loss of generality, we may assume that

$$\dot{x}^2(s) + \dot{y}^2(s) - \dot{r}^2(s) = \varepsilon, \quad \varepsilon = \pm 1, \tag{6.2}$$

which expresses the condition on s to be the arc-length of the meridian curve.

As a consequence of Proposition 3.4 we obtain:

Proposition 6.1. *A standard rotational surface given by* (6.1) *is immersed in* $\mathbb{E}^4_2$ *if and only if one of the following cases occurs:*

(i) $r(s)$ *is non-constant;*

(ii) $r(s)$ *is a non-zero constant and either* $x(s)$ *or* $y(s)$ *is non-constant.*

Convention: in the rest of this section, we assume that one of the conditions *(i)* or *(ii)* in Proposition 6.1 is satisfied.

If $\mathcal{M}$ is a standard rotational surface of elliptic type immersed in $\mathbb{E}^4_2$, then the tangent frame field of $\mathcal{M}$ is determined by the following vector fields:

$$Z_s = (\dot{x}, \dot{y}, \dot{r}\cos t, \dot{r}\sin t);$$
$$Z_t = (0, 0, -r\sin t, r\cos t).$$

The coefficients of the induced metric on $\mathcal{M}$ are given by:

$$E = \langle Z_s, Z_s\rangle = \varepsilon = \pm 1; \qquad F = \langle Z_s, Z_t\rangle = 0; \qquad G = \langle Z_t, Z_t\rangle = -r^2(s).$$

The function $\Delta = EG - F^2$ is expressed as $\Delta = -\varepsilon r^2(s)$. The second derivatives of $Z(s,t)$ are given by:

$$\begin{aligned} Z_{ss} &= (\ddot{x}, \ddot{y}, \ddot{r}\cos t, \ddot{r}\sin t)\,; \\ Z_{st} &= (0, 0, -\dot{r}\sin t, \dot{r}\cos t)\,; \\ Z_{tt} &= (0, 0, -r\cos t, -r\sin t)\,, \end{aligned}$$

from which we obtain:

$$\begin{aligned} &\langle \nabla_{Z_s} Z_s, Z_s\rangle = \langle \nabla_{Z_s} Z_t, Z_s\rangle = \langle \nabla_{Z_t} Z_t, Z_t\rangle = \langle \nabla_{Z_s} Z_s, Z_t\rangle = 0; \\ &\langle \nabla_{Z_t} Z_t, Z_s\rangle = -\langle \nabla_{Z_s} Z_t, Z_t\rangle = r\dot{r}. \end{aligned}$$

Let $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ be a vector field tangent to $\mathcal{M}$, α and β being arbitrary smooth functions of (s,t). Then, the Lie derivative $\mathcal{L}_\xi$ with respect to ξ is expressed by:

$$\begin{aligned} &\left(\mathcal{L}_\xi \langle , \rangle\right)(Z_s, Z_s) = 2\varepsilon\dot{\alpha}; \\ &\left(\mathcal{L}_\xi \langle , \rangle\right)(Z_s, Z_t) = \varepsilon\alpha' - \dot{\beta}r^2; \\ &\left(\mathcal{L}_\xi \langle , \rangle\right)(Z_t, Z_t) = -2(\alpha r\dot{r} + \beta' r^2). \end{aligned} \tag{6.3}$$

Theorem 6.2. *Let $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ be a Killing vector field on a standard rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$. Then either*

(i) $\alpha = 0$ and β is constant;

or

(ii) $r(s)$ satisfies the equation

$$\dot{r}^2 - r\ddot{r} = c \tag{6.4}$$

for some constant $c \in \mathbb{R}$ and the vector field ξ falls into one of the following cases:

case I. $\alpha = c_1 e^{\sqrt{c\varepsilon}\,t} + c_2 e^{-\sqrt{c\varepsilon}\,t}$; $\beta = -\frac{\dot{r}}{\sqrt{c\varepsilon}r}\left(c_1 e^{\sqrt{c\varepsilon}\,t} - c_2 e^{-\sqrt{c\varepsilon}\,t}\right) + c_3$, if $c\varepsilon > 0$;

case II. $\alpha = c_1 \cos\sqrt{-c\varepsilon}\,t + c_2 \sin\sqrt{-c\varepsilon}\,t$; $\beta = -\frac{\dot{r}}{\sqrt{-c\varepsilon}\,r}\left(c_1 \sin\sqrt{-c\varepsilon}\,t - c_2\cos\sqrt{-c\varepsilon}\,t\right) + c_3$, if $c\varepsilon < 0$;

case III. $\alpha = c_1 t + c_2$, if $c = 0$; and

sub-case III.1. r is a non-zero constant, $\beta = \frac{c_1\varepsilon}{r^2}s + c_3$;

sub-case III.2. $r(s) = pe^{qs}$ is non-constant, $\beta = -\frac{\dot{r}}{2r}\left(c_1 t^2 + 2c_2 t\right) - \frac{c_1\varepsilon}{2qp^2}e^{-2qs} + c_3$ for some constants $c_1, c_2, c_3, p, q \in \mathbb{R}$ with $c_1^2 + c_2^2 \neq 0$.

Proof. Using formulas (6.3), we get that ξ is Killing if and only if the following equalities hold:

$$\dot{\alpha} = 0; \qquad \varepsilon\alpha' = \dot{\beta}r^2; \qquad \beta' r = -\alpha\dot{r}. \tag{6.5}$$

Under the assumption that one of the conditions *(i)* or *(ii)* in Proposition 6.1 is satisfied, we have $r \neq 0$, so (6.5) can be written as

$$\dot{\alpha} = 0; \qquad \dot{\beta} = \frac{\varepsilon \alpha'}{r^2}; \qquad \beta' = -\frac{\alpha \dot{r}}{r}. \tag{6.6}$$

(i) If $\alpha = 0$, then system (6.6) implies $\beta = const$.

(ii) If $\alpha \neq 0$, then taking the derivative of the second equation in (6.6) with respect to t and of the third one with respect to s, we obtain

$$\varepsilon \alpha'' + \alpha(r\ddot{r} - \dot{r}^2) = 0.$$

Since $\alpha \neq 0$, from the last equality we get

$$\frac{\alpha''}{\alpha} = \dot{r}^2 - r\ddot{r},$$

which implies that both sides are equal to a constant $c \in \mathbb{R}$, i.e. (6.4) is fulfilled and

$$\alpha'' - c\varepsilon\alpha = 0. \tag{6.7}$$

Integrating the last equality, we obtain:

$$\begin{aligned} &\alpha = c_1 e^{\sqrt{c\varepsilon}\, t} + c_2 e^{-\sqrt{c\varepsilon}\, t}, \;\; \text{for } c\,\varepsilon > 0; \\ &\alpha = c_1 \cos\sqrt{-c\varepsilon}\, t + c_2 \sin\sqrt{-c\varepsilon}\, t, \;\; \text{for } c\,\varepsilon < 0; \\ &\alpha = c_1 t + c_2, \;\; \text{for } c = 0. \end{aligned} \tag{6.8}$$

The last equality of system (6.5) implies that β is determined by

$$\beta = -\frac{\dot{r}}{r} \int \alpha(t)dt + U(s) \tag{6.9}$$

for a smooth function $U(s)$. From the second equality of (6.5) it follows that $U(s)$ should satisfy

$$\frac{\dot{r}^2 - r\ddot{r}}{r^2} \int \alpha(t)dt + \dot{U}(s) = \frac{\varepsilon\alpha'}{r^2},$$

or equivalently

$$\frac{c}{r^2} \int \alpha(t)dt + \dot{U}(s) = \frac{\varepsilon\alpha'}{r^2}. \tag{6.10}$$

Now, using (6.8) we get

$$\varepsilon\alpha' - c \int \alpha(t)dt = \begin{cases} 0, \text{ if } c \neq 0 \\ \varepsilon c_1, \text{ if } c = 0 \end{cases} \tag{6.11}$$

Substituting (6.11) in (6.10), we obtain

$$\dot{U}(s) = \begin{cases} 0, \text{ if } c \neq 0 \\ \frac{\varepsilon c_1}{r^2}, \text{ if } c = 0 \end{cases}$$

which implies

$$U(s)=\begin{cases} c_3, \text{ if } c\neq 0\\ \varepsilon c_1\int \frac{ds}{r^2}, \text{ if } c=0\end{cases} \tag{6.12}$$

If $c\neq 0$, from (6.12), (6.9), and (6.8), we obtain that the function β is expressed by

$$\beta=-\frac{\dot r}{\sqrt{c\varepsilon}r}\left(c_1e^{\sqrt{c\varepsilon}\,t}-c_2e^{-\sqrt{c\varepsilon}\,t}\right)+c_3, \ \text{ if } \ c\varepsilon>0,$$

or

$$\beta=-\frac{\dot r}{\sqrt{-c\varepsilon}\,r}\left(c_1\sin\sqrt{-c\varepsilon}\,t-c_2\cos\sqrt{-c\varepsilon}\,t\right)+c_3, \ \text{ if } \ c\varepsilon<0,$$

which proves *case I* and *case II* of the theorem.

If $c=0$, from (6.4) we get the equation

$$\dot r^2-r\ddot r=0. \tag{6.13}$$

Since $r\neq 0$, the last equation implies

$$\dot r=qr \tag{6.14}$$

for some constant $q\in\mathbb{R}$. Then, the following sub-cases occur depending on $q=0$ or $q\neq 0$:

If r is a non-zero constant (i.e. $q=0$), then from (6.12) we obtain

$$U(s)=\frac{\varepsilon c_1}{r^2}\,s+c_3 \tag{6.15}$$

for some constant $c_3\in\mathbb{R}$. Having in mind that $\alpha=c_1t+c_2$, from (6.9) and (6.15) we obtain $\beta=\frac{c_1\varepsilon}{r^2}s+c_3$, which proves *sub-case III.1*.

If r is non-constant (i.e. $q\neq 0$), then from (6.14) we obtain $r=pe^{qs}$ for some $p\in\mathbb{R}$. In this case, using (6.12) we get

$$U(s)=-\frac{\varepsilon c_1}{2qp^2}\,e^{-2qs}+c_3, \tag{6.16}$$

where $c_3\in\mathbb{R}$. Again, having in mind that $\alpha=c_1t+c_2$, from (6.9) and (6.16) we obtain $\beta=-\frac{\dot r}{2r}\left(c_1t^2+2c_2t\right)-\frac{c_1\varepsilon}{2qp^2}e^{-2qs}+c_3$, which proves *sub-case III.2* and completes the proof of the theorem.

□

We recall from both differential geometry and physics, that on a semi-Riemannian manifold (N,g), a vector field ξ is called *homothetic*, if $\mathcal{L}_\xi g=2kg$ for some constant $k\in\mathbb{R}$. Moreover, ξ is *proper homothetic*, if ξ is homothetic but not Killing, i.e. $k\neq 0$. Homothetic vector fields have applications in the study of singularities in general relativity and they can also be used to generate new solutions for Einstein equations by similarity reduction [26].

Having in mind the importance of homothetic vector fields mentioned above, our purpose in the next statement, is to describe the feature of all proper homothetic vector fields on a standard rotational surface.

Theorem 6.3. *Let $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ be a proper homothetic vector field on a standard rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$. Then either*

(i) $r = q(s+l)^{1-\frac{c}{k}}$, $\alpha = k(s+l)$, and $\beta = ct + c_3$;

or

(ii) $r(s) = \pm\sqrt{q}s$ and the vector field ξ falls into one of the following cases:

I. $\alpha = ks + c_1 e^{\sqrt{q\varepsilon}\,t} + c_2 e^{-\sqrt{q\varepsilon}\,t}$; $\beta = -\frac{\varepsilon\sqrt{q\varepsilon}}{qs}\left(c_1 e^{\sqrt{q\varepsilon}\,t} + c_2 e^{-\sqrt{q\varepsilon}\,t}\right) + c_3$, if $q\varepsilon > 0$;

II. $\alpha = ks + c_1 \cos\sqrt{-q\varepsilon}\,t + c_2 \sin\sqrt{-q\varepsilon}\,t$; $\beta = -\frac{\varepsilon\sqrt{-q\varepsilon}}{qs}$ $(c_1 \cos\sqrt{-q\varepsilon}\,t + c_2 \sin\sqrt{-q\varepsilon}\,t) + c_3$, if $q\varepsilon < 0$;

for some constants $c, c_1, c_2, c_3, l, q \in \mathbb{R}$ with $q \neq 0$.

Proof. Using formulas (6.3), we get that ξ is proper homothetic if and only if the following equalities hold:

$$\dot{\alpha} = k; \qquad \varepsilon\alpha' = \dot{\beta}r^2; \qquad \alpha\dot{r} - \beta' r = kr,$$

where $k \neq 0$. Since $r \neq 0$, the above equalities imply

$$\alpha = ks + V(t); \qquad \dot{\beta} = \frac{\varepsilon V'}{r^2}; \qquad \beta' = k - \alpha\frac{\dot{r}}{r}, \tag{6.17}$$

for some smooth function $V(t)$. Taking the derivative in the second equation of (6.17) with respect to t and in the last one with respect to s, we obtain

$$\varepsilon V'' = -kr\dot{r} - (ks+V)(r\ddot{r} - \dot{r}^2), \tag{6.18}$$

whose derivative with respect to t gives:

$$\varepsilon V''' - V'(\dot{r}^2 - r\ddot{r}) = 0. \tag{6.19}$$

(i) If $V' = 0$, then $V(t) = kl = const$ and equalities (6.17) take the form:

$$\alpha = k(s+l); \qquad \dot{\beta} = 0; \qquad \beta' = k - k(s+l)\frac{\dot{r}}{r}.$$

The second of the above equalities implies $\beta = \beta(t)$, and hence from the third equality it follows that both its sides are equal to a constant $c \in \mathbb{R}$. Hence, we get $\beta(t) = ct + c_3$ and $(s+l)\frac{\dot{r}}{r} = 1 - \frac{c}{k}$, which implies $r = q(s+l)^{1-\frac{c}{k}}$.

(ii) If $V' \neq 0$, then (6.19) yields

$$\varepsilon\frac{V'''}{V'} = \dot{r}^2 - r\ddot{r},$$

which shows that both of its sides are equal to a constant q, i.e. $\dot{r}^2 - r\ddot{r} = q$. Then, from (6.18) we have

$$\varepsilon V'' - Vq = k(sq - r\dot{r}), \tag{6.20}$$

which has both sides equal to a constant $kp \in \mathbb{R}$. Hence,

$$sq - r\dot{r} = p, \tag{6.21}$$

whose derivative with respect to s is $q - \dot{r}^2 - r\ddot{r} = 0$. Having in mind that $\dot{r}^2 - r\ddot{r} = q$, we obtain $\dot{r}^2 = q$, and hence $q \geq 0$. Using (6.21), we get $r =$

$\pm\sqrt{s^2q-2ps}$. But $\dot{r}^2=q$, so we obtain $p=0$. Consequently, we get $r(s)=\pm\sqrt{q}s$, $q\neq 0$. From (6.20) and $p=0$ we obtain:

$$\varepsilon F''-Fq=0.$$

If $\varepsilon q>0$, then $V=c_1e^{\sqrt{q\varepsilon}\,t}+c_2e^{-\sqrt{q\varepsilon}\,t}$, which substituted into (6.17) gives us

$$\alpha=ks+c_1e^{\sqrt{q\varepsilon}\,t}+c_2e^{-\sqrt{q\varepsilon}\,t};\qquad \beta=-\frac{\varepsilon\sqrt{q\varepsilon}}{qs}\left(c_1e^{\sqrt{q\varepsilon}\,t}+c_2e^{-\sqrt{q\varepsilon}\,t}\right)+H(t)$$

for some smooth function $H(t)$. The last equality of (6.17) shows that $H(t)$ is constant. This proves case I of the theorem. Case II can be proved in a similar way.

□

7. Harmonicity on General Rotational Surfaces

The present section is devoted to the study of harmonic functions on general rotational surfaces. From now on, we assume that $\mathcal{M}$ is a non-degenerate general rotational surface in $\mathbb{E}^4_2$, i.e $\Delta\neq 0$.

A tangent frame field $\{e_1=Z_s,e_2\}$ of $\mathcal{M}$ is orthonormal, if

$$\langle e_1,e_1\rangle=\varepsilon;\quad \langle e_1,e_2\rangle=0;\quad \langle e_2,e_2\rangle=\varepsilon\sigma, \tag{7.1}$$

where $\sigma=\mathrm{sign}\Delta$. Obviously, $e_2=\pm\frac{1}{\sqrt{|\Delta|}}(FZ_s-\varepsilon Z_t)$. We deal here with the "+" case, the "−" case being similar. Using formulas (5.1), we obtain the following derivative formulas:

$$\begin{aligned}&\nabla_{Z_s}Z_s=-\tfrac{\sigma}{\sqrt{|\Delta|}}\dot{F}e_2; &&\nabla_{Z_s}Z_t=-\tfrac{\sigma}{2\sqrt{|\Delta|}}\dot{G}e_2;\\ &\nabla_{Z_t}Z_t=-\tfrac{\varepsilon\dot{G}}{2}\left(e_1+\tfrac{\sigma}{\sqrt{|\Delta|}}Fe_2\right).\end{aligned} \tag{7.2}$$

Then, for the orthonormal tangent frame field $\{e_1,e_2\}$ we obtain:

$$\nabla_{e_1}e_1=-\frac{\sigma}{\sqrt{|\Delta|}}\dot{F}e_2;\qquad \nabla_{e_2}e_2=-\frac{\sigma\dot{\Delta}}{2\Delta}e_1. \tag{7.3}$$

To fix notations, we recall that the *divergence* of a vector field X on a semi-Riemannian manifold (N,g) is given by

$$divX=\delta X=trace\langle\nabla.X,\cdot\rangle, \tag{7.4}$$

where ∇ denotes the Levi-Civita connection. In local coordinates (x^i), the divergence of the vector field $X=X^k\frac{\partial}{\partial x^k}$ is expressed by:

$$divX=\frac{\partial X^i}{\partial x^i}+X^i\,\Gamma^i_{ij}, \tag{7.5}$$

where the functions Γ^k_{ij} denote the Christoffel symbols of ∇.

Proposition 7.1. *Let $X = \alpha(s,t)Z_s + \beta(s,t)Z_t$ be a vector field tangent to a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$. Then, its divergence $divX$ is given by:*

$$divX = \dot{\alpha} + \beta' + \alpha\frac{\dot{\Delta}}{2\Delta}. \tag{7.6}$$

Proof. From (7.4) and (7.1) one has

$$divX = \varepsilon\langle\nabla_{e_1}X, e_1\rangle + \varepsilon\sigma\langle\nabla_{e_2}X, e_2\rangle,$$

which together with (7.1) and (7.3) yields (7.6) by direct computation.

□

A vector field X on a semi-Riemannian manifold is called *divergence free* if its divergence is zero, i.e. $divX = 0$.

As a consequence of formula (7.6), for the divergence of a vector field X on a general rotational surface $\mathcal{M}$, we obtain:

Corollary 7.2. *A vector field $X = \alpha(s,t)Z_s + \beta(s,t)Z_t$ tangent to a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is divergence free if and only if*

$$\beta = -\frac{\dot{\Delta}}{2\Delta}\int\alpha\,dt - \int\dot{\alpha}\,dt + U(s), \tag{7.7}$$

for a smooth function $U(s)$.

Corollary 7.3. *A vector field $X = \alpha(s,t)Z_s + \beta(s,t)Z_t$ tangent to a standard rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is divergence free if and only if*

$$\beta = -\frac{\dot{G}}{2G}\int\alpha\,dt - \int\dot{\alpha}\,dt + U(s), \tag{7.8}$$

for a smooth function $U(s)$.

Definition 7.4. On a semi-Riemannian manifold (N, g) with Levi-Civita connection ∇, the *co-differential* of a one-form ω is given by

$$\delta\omega = trace\,(\nabla.\omega)\cdot \tag{7.9}$$

The one-form ω is called *co-closed* (resp. *harmonic*), if $\delta\omega = 0$ (resp. $\delta\omega = 0 = d\omega$).

In local coordinates (x^i), the co-differential of ω is expressed by:

$$\delta\omega = g^{ij}\left(\frac{\partial\omega^i}{\partial x^j} - \Gamma^k_{ij}\,\omega_k\right). \tag{7.10}$$

On a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$, whose metric h is the restriction of $\langle.,.\rangle$ to $\mathcal{M}$, the Christoffel symbols Γ^k_{ij} with respect to the basis $\{Z_s, Z_t\}$ are given by:

$$\begin{aligned} &\Gamma^1_{11} = -\tfrac{F\dot{F}}{\Delta}; \qquad \Gamma^1_{12} = -\tfrac{F\dot{G}}{2\Delta}; \qquad \Gamma^1_{22} = -\tfrac{G\dot{G}}{2\Delta};\\ &\Gamma^2_{11} = \tfrac{\varepsilon\dot{F}}{\Delta}; \qquad \Gamma^2_{12} = \tfrac{\varepsilon\dot{G}}{2\Delta}; \qquad \Gamma^2_{22} = \tfrac{F\dot{G}}{2\Delta}; \end{aligned} \tag{7.11}$$

and the inverse of the metric h is given by:

$$h^{ij} = \frac{1}{\Delta}\begin{pmatrix} G & -F \\ -F & \varepsilon \end{pmatrix}. \tag{7.12}$$

Proposition 7.5. *Let $\omega = \alpha(s,t)\,ds + \beta(s,t)\,dt$ be a one-form on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$. Then, the co-differential $\delta\omega$ is given by:*

$$\delta\omega = \frac{\alpha}{\Delta^2}\left(FfG - F^2\dot{G} + \varepsilon Gg\right) + \frac{\dot{\alpha}G}{\Delta} + \beta\varepsilon\frac{Fg - Gf}{\Delta^2} - \frac{F}{\Delta}(\alpha' + \dot{\beta}) + \frac{\varepsilon\beta'}{\Delta}. \tag{7.13}$$

Proof. Using local coordinates, from (7.10) we have:

$$\begin{aligned} \delta\omega = {} & \tfrac{G}{\Delta}\left(\dot{\alpha} - \Gamma^1_{11}\alpha - \Gamma^2_{11}\beta\right) - \tfrac{F}{\Delta}\left(\dot{\beta} - \Gamma^1_{12}\alpha - \Gamma^2_{12}\beta\right) - \\ & - \tfrac{F}{\Delta}\left(\alpha' - \Gamma^1_{21}\alpha - \Gamma^2_{21}\beta\right) + \tfrac{\varepsilon}{\Delta}\left(\beta' - \Gamma^1_{22}\alpha - \Gamma^2_{22}\beta\right). \end{aligned} \tag{7.14}$$

Then (7.13) follows from (7.11) and (7.14) by direct computation. □

As a consequence of Proposition 7.5 we obtain the following statement.

Corollary 7.6. *A one-form $\omega = \alpha(s,t)\,ds + \beta(s,t)\,dt$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is:*

(i) co-closed if and only if

$$\alpha(FfG - F^2\dot{G} + \varepsilon Gg) + \dot{\alpha}G\Delta + \beta\varepsilon(Fg - Gf) - F\Delta(\alpha' + \dot{\beta}) + \beta'\varepsilon\Delta = 0; \tag{7.15}$$

(ii) harmonic if and only if (7.15) and $\dot{\beta} = \alpha'$ hold true.

Having in mind the expressions for the coefficients of the first fundamental form of a standard rotational surface, from Corollary 7.6, we obtain:

Corollary 7.7. *A one-form $\omega = \alpha(s,t)\,ds + \beta(s,t)\,dt$ on a standard rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is:*

(i) co-closed if and only if

$$\alpha = \frac{1}{r}\left(\varepsilon\int\frac{\beta'}{r}\,ds + U(t)\right), \tag{7.16}$$

for a smooth function $U(t)$;

(ii) harmonic if and only if (7.16) is satisfied with $U(t) = \int r\dot{\beta}\,dt - \varepsilon\int\frac{\beta'}{r}\,ds$.

For the following well known notion, see for instance [27], pp.435-438.

Definition 7.8. On a semi-Riemannian manifold (N, g) a smooth function φ is *harmonic*, if it satisfies one of the following equivalent conditions:

(i) $d\varphi$ is harmonic;

(ii) $d\varphi$ is co-closed;

(iii) $\bar{\Delta}\varphi = \sum_i g(e_i, e_i)\,(e_i e_i \varphi - \nabla_{e_i} e_i \varphi) = 0$ for any pseudo-orthonormal frame $\{e_i\}_i$, where $\bar{\Delta}$ denotes the Laplacian of φ;

(iv) $g^{ij}\left(\frac{\partial^2\varphi}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial\varphi}{\partial x^k}\right) = 0$ in local coordinates (x^i);

(v) $div(grad\,\varphi) = 0$, where $grad\,\varphi$ is defined as the dual vector field of the one-form $d\varphi$ with respect to g, i.e. $d\varphi = g(grad\,\varphi, \cdot)$.

From Corollary 7.6 we obtain

Proposition 7.9. *A smooth function $\varphi(s,t)$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is harmonic if and only if*

$$G\ddot{\varphi} - 2F\dot{\varphi}' + \varepsilon\varphi'' + \frac{1}{\Delta}\left(fF - F^2G + \varepsilon Gg\right)\dot{\varphi} + \frac{\varepsilon}{\Delta}\left(gF - fG\right)\varphi' = 0. \quad (7.17)$$

Remark 7.10. Equation (7.17) is never parabolic, since $\Delta \neq 0$. It is either elliptic or hyperbolic according to $\sigma > 0$ or $\sigma < 0$.

Proposition 7.11. *A smooth function $\varphi(s,t)$ on a standard rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is harmonic if and only if*

$$r^2\ddot{\varphi} - \varepsilon\varphi'' = \frac{\dot{G}}{2}\,\dot{\varphi}. \quad (7.18)$$

Remark 7.12. Equation (7.18) is either elliptic or hyperbolic according to $\varepsilon = -1$ or $\varepsilon = 1$.

8. Special Vector Fields on General Rotational Surfaces

Definition 8.1. On a semi-Riemannian manifold (N, g), a vector field ξ is called *geodesic* (resp. *parallel*), if

$$\nabla_\xi \xi = \varphi\,\xi \quad (8.1)$$

for a smooth function φ called the potential function of ξ (resp. $\varphi = 0$).

Example 8.2. On a gradient Yamabe soliton, the soliton field is a geodesic field.

Indeed, on a gradient Yamabe soliton (M, g, ξ, λ), the soliton field ξ satisfies

$$\nabla_\xi \xi = (S - \lambda)\xi, \quad (8.2)$$

where S is the scalar curvature and λ is a constant [28]. Equality (8.2) implies that ξ is a geodesic field with potential function $\varphi = S - \lambda$.

The following statements follow from Definition 8.1 and formulas (7.11).

Proposition 8.3. *A vector field $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ tangent to a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is geodesic if and only if the following conditions are satisfied:*

$$\alpha\dot{\alpha} + \beta\alpha' - \frac{1}{2\Delta}(2\alpha^2 F\dot{F} + 2\alpha\beta F\dot{G} + \beta^2 G\dot{G}) = \varphi\alpha; \quad (8.3)$$

$$\alpha\dot{\beta} + \beta\beta' + \frac{1}{2\Delta}(2\alpha^2\varepsilon\dot{F} + 2\alpha\beta\varepsilon\dot{G} + \beta^2 F\dot{G}) = \varphi\beta; \tag{8.4}$$

for a smooth function $\varphi(s,t)$.

Proposition 8.4. *A vector field $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ tangent to a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is parallel if and only if the following conditions are satisfied:*

$$\alpha\dot{\alpha} + \beta\alpha' - \frac{1}{2\Delta}(2\alpha^2 F\dot{F} + 2\alpha\beta F\dot{G} + \beta^2 G\dot{G}) = 0; \tag{8.5}$$

$$\alpha\dot{\beta} + \beta\beta' + \frac{1}{2\Delta}(2\alpha^2\varepsilon\dot{F} + 2\alpha\beta\varepsilon\dot{G} + \beta^2 F\dot{G}) = 0. \tag{8.6}$$

Corollary 8.5. *A vector field $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ tangent to a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ with orthogonal parametric lines is:*

(i) geodesic if and only if the following conditions are satisfied:

$$\alpha\dot{\alpha} + \beta\alpha' - \frac{\beta^2\dot{G}}{2} = \varphi\alpha; \tag{8.7}$$

$$\alpha\dot{\beta} + \beta\beta' + \alpha\beta\frac{\dot{G}}{G} = \varphi\beta; \tag{8.8}$$

for a smooth function $\varphi(s,t)$;

(ii) parallel if and only if the following conditions are satisfied:

$$\alpha\dot{\alpha} + \beta\alpha' - \frac{\beta^2\dot{G}}{2} = 0; \tag{8.9}$$

$$\alpha\dot{\beta} + \beta\beta' + \alpha\beta\frac{\dot{G}}{G} = 0. \tag{8.10}$$

Corollary 8.6. *A curve $\gamma(u) = Z(s(u),t(u))$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is a geodesic if and only if the following conditions hold:*

$$2s''\Delta - (2s'^2 F\dot{F} + 2s't'F\dot{G} + t'^2 G\dot{G}) = 0; \tag{8.11}$$

$$2t''\Delta + 2\varepsilon s'^2\dot{F} + 2\varepsilon s't'\dot{G} + t'^2 F\dot{G} = 0; \tag{8.12}$$

where s' and t' denote the derivative of $s(u)$ and $t(u)$ with respect to u, respectively.

Corollary 8.7. *The meridian curve (3.1) on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is a geodesic if and only if the function F is constant.*

From Corollary 8.6 we obtain:

Corollary 8.8. *A curve $\gamma(u) = Z(s(u),t(u))$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ with orthogonal parametric lines is a geodesic if and only if either*

(i) $s = c_1 u + c_2$ and t is constant,

or

(ii) $s = \int\sqrt{c_2 - \frac{\varepsilon c_1^2}{G(s(u))}}\,du$; $\quad t = \int\frac{c_1}{G(s(u))}\,du$;

for some constants $c_1, c_2 \in \mathbb{R}$.

Proof. From (8.11) and (8.12) one obtains respectively:

$$2s''\varepsilon - t'^2\dot{G} = 0; \tag{8.13}$$

$$t''G + s't'\dot{G} = 0. \tag{8.14}$$

If $t' = 0$, then t is constant and (8.13) implies $s = c_1u + c_2$. Otherwise, if $t' \neq 0$, then (8.14) can be written as:

$$\frac{d}{du}\left(\ln(t') + \ln(G(s(u)))\right) = 0,$$

which gives $t'G = c_1$ with $c_1 \in \mathbb{R}$ and hence, $t = \int \frac{c_1}{G(s(u))}\,du$. By replacing t into (8.13), one obtains

$$s'' = \frac{\varepsilon c_1^2}{2G^2}\,\dot{G}.$$

Multiplying both sides of the last equality with $2s'$, it follows that:

$$\frac{d}{du}\left(s'^2 + \frac{\varepsilon c_1^2}{G(s(u))}\right) = 0,$$

from which we deduce $s = \int \sqrt{c_2 - \frac{\varepsilon c_1^2}{G(s(u))}}\,du$ and complete the proof.

□

Definition 8.9. A vector field ξ on a semi-Riemannian manifold (N, g) is called *concircular*, if there exists a function λ on N such that

$$\nabla_X \xi = \lambda X, \qquad \forall\, X \in \Gamma(TN), \tag{8.15}$$

where ∇ is the Levi-Civita connection [29]. In particular, if the concircular vector field ξ admits a smooth function φ on N such that $\xi = grad\,\varphi$, then φ is called a *concircular function* (or a *concircular scalar field*).

Concircular vector fields play an important role in the theory of conformal transformations [30]. Such vector fields have also interesting applications in physics [31].

Proposition 8.10. *A vector field $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is concircular, if and only if there exists a smooth function $\lambda(s,t)$ on $\mathcal{M}$ such that the following conditions are satisfied:*

$$\dot{\alpha} + \varepsilon F\dot{\beta} = \lambda; \tag{8.16}$$

$$\alpha' - \frac{\dot{G}}{2\Delta}(\alpha F + \beta G) = 0; \tag{8.17}$$

$$\dot{\beta} + \frac{\varepsilon}{2\Delta}(2\alpha\dot{F} + \beta\dot{G}) = 0; \tag{8.18}$$

$$\beta' + \frac{\dot{G}}{2\Delta}(\alpha\varepsilon + \beta F) = \lambda. \tag{8.19}$$

Proof. From (5.3) and (8.15) it follows that ξ is concircular, if and only if the following relations hold true:

$$\langle \nabla_{Z_s}\xi, Z_s\rangle = \lambda\varepsilon; \qquad \langle \nabla_{Z_s}\xi, Z_t\rangle = \langle \nabla_{Z_t}\xi, Z_s\rangle = \lambda F; \qquad \langle \nabla_{Z_t}\xi, Z_t\rangle = \lambda G.$$

These relations are equivalent to (8.16) together with the following three relations:

$$\dot{\alpha}F + \alpha\dot{F} + \dot{\beta}G + \beta\frac{\dot{G}}{2} = \lambda F; \tag{8.20}$$

$$\alpha'\varepsilon + \beta' F - \beta\frac{\dot{G}}{2} = \lambda F; \tag{8.21}$$

$$\alpha' F + \alpha\frac{\dot{G}}{2} + \beta' G = \lambda G. \tag{8.22}$$

Then, (8.16) and (8.20) yield (8.18), and from (8.21) and (8.22) one obtains (8.17) and (8.19), which complete the proof. $\square$

Corollary 8.11. *A smooth function $\varphi(s,t)$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ is concircular, if and only if there exists a smooth function $\lambda(s,t)$ on $\mathcal{M}$ such that the following conditions are satisfied:*

$$\ddot{\varphi} + \varepsilon F\ddot{\beta} = \lambda; \tag{8.23}$$

$$\dot{\varphi}' - \frac{\dot{G}}{2\Delta}(\dot{\varphi}F + \varphi' G) = 0; \tag{8.24}$$

$$\dot{\varphi}' + \frac{\varepsilon}{2\Delta}(2\dot{\varphi}\dot{F} + \varphi'\dot{G}) = 0; \tag{8.25}$$

$$\varphi'' + \frac{\dot{G}}{2\Delta}(\dot{\varphi}\varepsilon + \varphi' F) = \lambda. \tag{8.26}$$

Corollary 8.12. *A vector field $\xi = \alpha(s,t)Z_s + \beta(s,t)Z_t$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ with orthogonal parametric lines is concircular, if and only if*

$$\alpha = \varepsilon\dot{\sqrt{G}}\,k(t) + h(s); \tag{8.27}$$

$$\beta = \frac{k'(t)}{\sqrt{G}}, \tag{8.28}$$

where $k(t)$ and $h(s)$ are smooth functions belonging to one of the following cases:

Case I: k is constant and $h = q_1\sqrt{G} - \varepsilon k\dot{\sqrt{G}}$ for some constant $q_1 \in \mathbb{R}$.

Case II: k is non-constant, $h = \sqrt{G}\int \frac{q_2}{G}ds$ for some $q_2 \in \mathbb{R}$ and $k(t)$ is expressed in one of the following ways:

(i) $k(t) = c_1 e^{\sqrt{q_1}\,t} + c_2 e^{-\sqrt{q_1}\,t} - q_2$, for $q_1 > 0$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$;

(ii) $k(t) = c_1\cos\sqrt{-q_1}\,t + c_2\sin\sqrt{-q_1}\,t - q_2$, for $q_1 < 0$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$;

(iii) $k(t) = q_2\frac{t^2}{2} + c_1 t + c_2$, where $c_1, c_2 \in \mathbb{R}$, $c_1^2 + q_2^2 \neq 0$.

Proof. In the case of orthogonal parametric lines (i.e. $F = 0$), from Proposition 8.10 we have that ξ is concircular if and only if the following conditions are satisfied:

$$\dot{\alpha} = \lambda; \tag{8.29}$$

$$\alpha' = \frac{\varepsilon \dot{G}}{2}\beta; \tag{8.30}$$

$$\dot{\beta} = -\frac{\dot{G}}{2G}\beta; \tag{8.31}$$

$$\beta' = \lambda - \alpha\frac{\dot{G}}{2G}. \tag{8.32}$$

It follows from (8.31) that $\beta = \frac{k'(t)}{\sqrt{G}}$ for some smooth function $k(t)$. Then, using (8.30) we obtain $\alpha = \varepsilon \dot{\sqrt{G}}\, k(t) + h(s)$ for some smooth function $h(s)$, i.e. (8.27) is fulfilled. Now, from (8.29), (8.32), and (8.27), we get

$$k'' = \frac{\varepsilon}{2}\,\frac{\ddot{G}G - \dot{G}^2}{G}\,k + G\frac{d}{ds}\left(\frac{h}{\sqrt{G}}\right), \tag{8.33}$$

which shows that one of the following cases may occur:

Case I: If k is constant, then (8.33) can be written as

$$\frac{d}{ds}\left(\frac{h}{\sqrt{G}} + \frac{\varepsilon k}{2}\frac{\dot{G}}{G}\right) = 0,$$

or equivalently $h = q_1\sqrt{G} - \varepsilon k \dot{\sqrt{G}}$ for some constant $q_1 \in \mathbb{R}$.

Case II: If k is non-constant, then in (8.33) both the coefficient of k and the last term in the right-hand side, which doesn't contain k, should be constant, i.e.

$$k'' = q_1 k + q_2, \tag{8.34}$$

which implies that $\frac{\ddot{G}G-\dot{G}^2}{G} = 2\varepsilon q_1$ and $h(s) = \sqrt{G}\int \frac{q_2}{G} ds$ should be satisfied. Then, (8.34) yields one of the subcases *(i)*, *(ii)*, *(iii)* according to $q_1 > 0$, $q_1 < 0$, $q_1 = 0$, which complete the proof.

□

Corollary 8.13. *A smooth function $\varphi(s,t)$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ with orthogonal parametric lines is concircular, if and only if one of the following cases occurs:*

(i) G is constant and $\varphi(s,t) = \lambda\frac{s^2+t^2}{2} + c_1 s + c_2 t + c_3$ for $\lambda = const$ and $c_1, c_2, c_3 \in \mathbb{R}$;

(ii) The function φ is given by

$$\varphi = c_1 \int \sqrt{G} ds + c_2; \tag{8.35}$$

for some constants $c_1, c_2 \in \mathbb{R}$.

Proof. Applying Corollary 8.11 in the case of orthogonal parametric lines, the relations (8.23) – (8.26) become respectively:

$$\ddot{\varphi} = \lambda; \tag{8.36}$$

$$\dot{\varphi}' = \frac{\varepsilon \dot{G}}{2} \varphi'; \tag{8.37}$$

$$\dot{\varphi}' = -\frac{\dot{G}}{2G} \varphi'; \tag{8.38}$$

$$\varphi'' = \lambda - \frac{\dot{G}}{2G} \dot{\varphi}, \tag{8.39}$$

for a smooth function $\lambda(s,t)$. It follows from (8.37) and (8.38) that either $\dot{G} = 0$ or $\varphi' = 0$. We consider separately these two cases.

(i) If G is constant, then (8.37) and (8.38) imply $\varphi(s,t) = U(s) + V(t)$ for some smooth functions $U(s)$ and $V(t)$. Then, from (8.36) and (8.39), we obtain that λ is constant and φ is expressed by $\varphi(s,t) = \lambda \frac{s^2+t^2}{2} + c_1 s + c_2 t + c_3$, $c_1, c_2, c_3 \in \mathbb{R}$.

(ii) If $\varphi' = 0$, i.e. $\varphi = \varphi(s)$ is independent of t, then, from (8.36) and (8.39) we obtain

$$\ddot{\varphi} = \frac{\dot{G}}{2G} \dot{\varphi}, \tag{8.40}$$

which implies either $\dot{\varphi} = 0$ (and hence φ is constant and (8.35) is satisfied for $c_1 = 0$), or $\dot{\varphi} \neq 0$ (from which we get (8.35) by dividing (8.40) by $\dot{\varphi}$). This completes the proof.

□

At the end, we focus our attention on concurrent vector fields.

Definition 8.14. A vector field ξ on a semi-Riemannian manifold (N, g) is called *concurrent*, if (8.15) is satisfied for $\lambda = 1$.

Riemannian manifolds equipped with concurrent vector fields have been intensively studied by many mathematicians (see, e.g. [32–35]).

From Corollary 8.12 we obtain:

Corollary 8.15. *A vector field $\xi = \alpha(s,t) Z_s + \beta(s,t) Z_t$ on a general rotational surface $\mathcal{M}$ in $\mathbb{E}^4_2$ with orthogonal parametric lines is concurrent, if and only if one of the following cases occurs:*

(i) the functions G, α, β are given respectively by

$$G = \left(\frac{s + q_0}{q_1} \right)^2; \qquad \alpha = s + q_0; \qquad \beta = 0,$$

for some constants $q_0, q_1 \in \mathbb{R}$, with $q_1 \neq 0$;

(ii) the functions G, α, β are given respectively by

$$G = (cs + c_0)^2; \qquad \alpha = \varepsilon c\, k + s + q_0; \qquad \beta = \frac{k'}{cs + c_0},$$

for some constants $q_0, c, c_0 \in \mathbb{R}$*, such that* G *is non-zero and* $k(t)$ *is a smooth function given by:*

$k(t) = c_1 e^{\sqrt{q_1}\,t} + c_2 e^{-\sqrt{q_1}\,t} - q_2$*, for* $q_1 > 0$*, where* $q_1 = -\varepsilon c^2$*;*

$k(t) = c_1 \cos\sqrt{-q_1}\,t + c_2 \sin\sqrt{-q_1}\,t - q_2$*, for* $q_1 < 0$*, where* $q_1 = -\varepsilon c^2$*;*

$k(t) = q_2 \frac{t^2}{2} + c_1 t + c_2$*, for* $c = 0$*.*

Remark. The expressions of G in *(i)* and *(ii)* are not equivalent, since the constant c may be zero.

Each of the Corollaries 8.13 and 8.15 yields the following

Corollary 8.16. *A smooth function* $\varphi(s,t)$ *on a general rotational surface* $\mathcal{M}$ *in* $\mathbb{E}^4_2$ *with orthogonal parametric lines has a concurrent gradient vector field, if and only if one of the following cases occurs:*

(i) G *is a non-zero constant and* $\varphi(s,t) = \frac{s^2+t^2}{2} + c_1 s + c_2 t + c_3$*, for some constants* $c_1, c_2, c_3 \in \mathbb{R}$*;*

(ii) φ *is given by* $\varphi = c_1 \int \sqrt{G} ds + c_2$*, for some constants* $c_1, c_2 \in \mathbb{R}$*.*

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